Homework 6

Due: March 12, 2025

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problems marked with a * are problems which may appear on the midterm or final with some modification.

Problem 1

For each of the statements below, determine if it is *always* true, *sometimes* true, or *never* true. Justify your answers. To justify an "always" or "never" answer, write a proof; to justify a "sometimes" answer, give one witness that makes the statement true and one that makes the statement false, explaining these judgments.*

For example, the statement

Let $a, b : \mathbb{N}$ and suppose $a \mid b$. Then the greatest prime factor of b divides a.

is sometimes true. It is true if a = 6 and b = 12, since $6 \mid 12$ and the greatest prime factor of 12 is 3, which divides 6. It is false if a = 2 and b = 6, since $2 \mid 6$ but the greatest prime factor of 6 is 3, which does not divide 2.

- a. Let $p, q, r, s : \mathbb{N}$ be prime numbers and suppose that pq = rs. Then p = r and q = s.
- b. Let $p : \mathbb{N}$ be prime. Then p is relatively prime to every positive natural number except for p itself.
- c. Let $a, b, c, n : \mathbb{N}$ and suppose that $3ab \equiv 3ac \mod n$. Then $b \equiv c \mod n$.
- d. Let $a, b, m, n : \mathbb{N}$ be larger than 1 where $n \mid m$ and $a \equiv b \mod m$. Then $a \equiv b \mod n$.
- e. Let $a, b : \mathbb{N}$. Then gcd(1 + a, 1 + b) = 1 + gcd(a, b).
- f. Let $a, b, c, d, n : \mathbb{N}$ be integers with c and d positive and $n \ge 2$. If $a \equiv b \mod n$ and $c \equiv d \mod n$ then $a^c \equiv b^d \mod n$.

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Problem 2

For each of the following, find the multiplicative inverse for the given element by using the extended Euclidean algorithm. If no inverse exists, explain why. *

a. 4 (mod 17)b. 25 (mod 21)c. 4 (mod 6)

For each of the following, find the positive integer values for x that satisfy the congruence. If x has finitely many solutions, list all of them. If x has infinitely many solutions, state that there are infinitely many solutions and list three of them. *

d. $x \equiv 3 \pmod{4}$ e. $2x \equiv 7 \pmod{2}$ f. $2 \equiv 6 \pmod{x}$

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Problem 3

a. In a true mélange of classic animal-based math problems, a tortoise and hare are playing a game that involves hopping around a circular track of lily pads. At the start, they stand on the same lily pad. Then, every second, the slow tortoise jumps one lily pad clockwise, while the swift hare jumps two lily pads clockwise. They keep hopping until they once again end up on the same lily pad as each other (regardless of whether it is the lily pad on which they started).

If there are n lily pads, where n is a positive natural number, determine, with proof, the number of seconds it will take for the tortoise and have to finish their game.

HINT: Label the lify pads 0 through n - 1. Can you write down a function t(k) that outputs the number of the lify pad occupied by the tortoise after k seconds? What about h(k) for the hare? When are these functions equal?

As an example, here's how the game would go on a track of four lily pads (where t is the number of seconds elapsed, and T represents the tortoise and H the hare):



b. After completing their game, the tortoise and hare decide to play again on a different track of lily pads. To get to that track, they'll need to hop down a short road, which is conveniently made of an even number of lily pads.

The tortoise and hare both start at the first lily pad on the road. They both hop down the road, then begin hopping clockwise around the new track once they reach it. Once either animal is on the track, it continues hopping circularly around the track and never returns to the road. As before, the tortoise and hare keep hopping until they end up one the same lily pad. The setup is depicted below with the road in blue and the track in red:



As before, the tortoise hops one lily pad each second, while the hare hops two lily pads per second.

Let $r \in \mathbb{N}^+$ be the number of lily pads on the road, and let $c \in \mathbb{N}^+$ be the number of lily pads on the circular track. Show that, if $c \ge r$, the tortoise and hare will meet when exactly c seconds have elapsed and not before.

As an example, letting T denote the tortoise and H the hare, here's how this would play out with r = 2 and c = 4:





⁷ Problem 4 (Mind Bender — *Extra Credit*)

You are playing a game with a hare in which you and the hare alternate turns. The hare starts at 0 and each time it is the hare's turn, it jumps some non-zero distance $b : \mathbb{N}$ forward.

Each time it is your turn you can do one of two things: you can either (1) do nothing or (2) use your length $a : \mathbb{N}$ lasso to pull the hare distance a backwards. This is a very useful lasso which satisfies the property that a > b and gcd(a, b) = 1. You are also allowed to pull the hare to negative positions.

Show that for any $n : \mathbb{N}$, there is a series of actions (waits or pulls) you can always choose so that the hare is at n at some point.

For example, if a = 9, b = 4 and n = 2 and the game proceeds as follows, the hare is at 2 at some point: **Hare:** jumps to 4; **You:** pull the hare to -5; **Hare:** jumps to -1; **You:** pull the hare to -10; **Hare:** jumps to -6; **You:** do nothing; **Hare:** jumps to -2; **You:** do nothing; **Hare:** jumps to 2.