Homework 5

Due: Wednesday, March 6, 2024

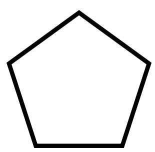
All homeworks are due at 11:59 PM on Gradescope.

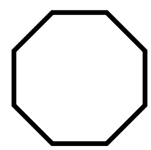
Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

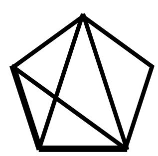
Problem 1

A *regular n-gon* is a polygon with n sides, all of equal length, and n angles, all of equal measure. For example, a square is a regular 4-gon, and the images below are a regular 5-gon and 8-gon:





A *diagonal* of an n-gon is a line connecting two non-adjacent vertices. For instance, here are three diagonals of the regular 5-gon:



Show using induction that for all $n \in \mathbb{N}$ where $n \geq 3$, a regular n-gon always has $\frac{(n)(n-3)}{2}$ diagonals.

Problem 2

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function with the property that $f(x + y) = f(x) + f(y)$ for all $x, y : \mathbb{R}$. (A function with this property is said to be *linear*; this word has appeared before in 22, in the context of *linear combinations*!)

a. Prove that there is a real number c such that for every $n \in \mathbb{N}$, $f(n) = c \cdot n$.

> **HINT:**
> You'll want to use induction here. But be careful with the order of quantifiers in the statement!

b. Extend this to the positive rational numbers: show that for any $a, b \in \mathbb{Z}^+$, $f\left(\frac{a}{b}\right) = c \cdot \frac{a}{b}$.

> **HINT:** What is $f(b/b)$?

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Homework05.lean` in the directory browser on the left of your screen in your Codespace. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, download the file to your computer (right-click on the file in the Codespace directory browser and click "Download"), and upload it to Gradescope.

Problem 4 (Mind Bender — *Extra Credit*)

Let $\langle a_k \rangle_{k \in \mathbb{N}}$ be a the sequence of natural numbers defined as follows:

- $a_0 = 0$.
- $a_1 = 1$.
- For all natural $k \geq 2$, $a_k = 2a_{k-1} + a_{k-2}$.

a. Show that for all $n \in \mathbb{N}$ such that $n \geq 1$, we have $\sum_{k=0}^{n} a_k < 2a_n$.

b. Show that we can write any number $n \in \mathbb{N}$ as the sum/difference of elements of the sequence, i.e., $n = a_{j_1} \pm a_{j_2} \pm \cdots \pm a_{j_r}$ for some distinct indices $j_1, j_2, \ldots, j_r$.

   (More formally, we are asking you to prove that for any $n \in \mathbb{N}$, there exist some

   - number of terms $r \in \mathbb{N}$,
   - indices $\{j_s \in \mathbb{N} \mid s \in \mathbb{N} \text{ and } s < r\}$, and
   - exponents $\{p_s \in \{0, 1\} \mid s \in \mathbb{N} \text{ and } s < r\}$

   for which $n = \sum_{s=0}^{r-1} (-1)^{p_s} a_{j_s}$.)

   You may cite without proof the fact that the sequence is strictly increasing for $k \geq 1$. You may also make use of your result in the preceding part.

HINT: Before trying to prove this, pick some particular values of n and try to write them as sums/differences of the terms of the sequence, and look for patterns as you do so. (You might find it useful to pick consecutive values of n.) In particular, consider whether each natural number n is less than, greater than, or equal to the sum of all sequence elements strictly less than it. How does this relate to whether you need to add terms or subtract them to obtain n?