Homework 4

Due: Wednesday, February 28, 2024

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

For each of the following relations, determine whether it is *reflexive*, whether it is *symmetric*, and whether it is *transitive*. Be sure to justify your answers. (If a relation has all these properties, then it is an *equivalence relation*.)

Reminder: for $x, y \in \mathbb{N}$, we say that x *divides* y, written $x|y$, if there exists $c \in \mathbb{N}$ such that $c \cdot x = y$.

a. Let R be the relation on $\mathbb{N}$ defined by the set of ordered pairs:

$$\{(a, b) \mid \exists d : \mathbb{N}, d > 1 \wedge \neg \mathsf{Prime}(d) \wedge d|a \wedge d|b\}.$$

b. Let F be the set of formulas of propositional logic: for example, $p \wedge q \in F$, $p \to (p \to r) \in F$, ...

   Let $R(\varphi, \psi)$ be the relation on F that holds when $\varphi \to \psi$ is provable.

c. Let L be the set of straight lines in a two-dimensional plane.

   Let R be the relation on L defined by the set of ordered pairs:

$$\{(\ell, m) \mid \ell \text{ and } m \text{ are parallel}\}.$$

   (Two lines are parallel if and only if they have the same slope in the Cartesian plane.)

d. Let $S = \{a, b, c\}$. Let R be the relation on S with the graph:

$$\{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c), (b, a), (c, b)\}.$$

e. At least one of the above relations is an equivalence relation. Choose one, and describe the *equivalence classes* of that relation.

Problem 2

Given sets A and B, a function $f : A \to B$ is *injective* if

$$\forall a_1, a_2 \in A,\ f(a_1) = f(a_2) \to a_1 = a_2.$$

The function f is *surjective* if

$$\forall b \in B, \exists a \in A, f(a) = b.$$

This is a formalization of the intuitive "arrow counting" definitions given in lecture.

We defined the following in recitation: given sets A, B, and C, and functions $g : B \to C$ and $f : A \to B$, the *composition* of g and f, written $g \circ f : A \to C$, is defined by $(g \circ f)(x) = g(f(x))$ for $x \in A$. In other words, to apply $g \circ f$ to an argument x, first apply f to x, and then apply g to the result. (Beware the order of operations!) Note that you can write the $\circ$ symbol in LaTeX with `\circ`.

a. Let A, B, C be sets and $g : B \to C$ and $f : A \to B$ be functions. Prove that if the composition $g \circ f$ is bijective, then f is injective and g is surjective. Be precise: structure your argument carefully, and think about the formal statements of injectivity and surjectivity.

b. Let A, B, C be sets and $g : B \to C$ and $f : A \to B$ be functions. If f is injective and g is surjective, is $g \circ f$ necessarily bijective? If so, prove this. If not, provide a counterexample: that is, give an example of three sets A, B, C, an injection $f : A \to B$, and a surjection $g : B \to C$ such that $g \circ f$ is *not* bijective. Explain why your examples of f and g are injective and surjective respectively, and why $g \circ f$ is not bijective.

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Homework04.lean` in the directory browser on the left of your screen in your Codespace. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, download the file to your computer (right-click on the file in the Codespace directory browser and click "Download"), and upload it to Gradescope.

Problem 4 (Mind Bender — *Extra Credit*)

Recall from earlier that, given sets A and B, a function $f : A \to B$ is *injective* if, for all $a, b \in A$, we have $f(a) = f(b) \to a = b$.

Here's another property of functions we can formally define: given a set A, we define the following property of a function $f : A \times A \to A$:

$$\text{For all } a, b, c \in A,\ f(a, f(b, c)) = f(f(a, b), c). \tag{$\star$}$$

Let A be a set. Suppose there exists an injective function $f : A \times A \to A$ with property $(\star)$. Determine, with proof, all possible values of $|A|$.