Homework 3

Due: Friday, February 23, 2024

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Let $A = \{0, 10, 20\}$, $B = \{\text{T-rex}, \text{brachiosaurus}\}$, $C = \{k : \mathbb{Z} \mid k/2 \in \{10, 20, 30\}\}$, $D = \{\}$, and $E = \mathbb{N}^+$.

Find the cardinalities of the following sets. Justify your answers.

a. $A \cap C$

b. $B \cup C$

c. $C \setminus A$

d. $B \cup D$

e. $C \times D$

f. $C \times A$

g. $D \cap E$

h. $\mathcal{P}(A)$

i. $\mathcal{P}(D)$

j. $\mathcal{P}(\mathcal{P}(A) \times C)$

Solution:

a. 1. $A \cap C = \{20\}$, since among the elements of A, $20/2 \in \{10, 20, 30\}$ while $0/2 \notin \{10, 20, 30\}$ and $10/2 \notin \{10, 20, 30\}$.

b. 5. $B \cup C = \{\text{T-rex}, \text{brachiosaurus}, 20, 40, 60\}$. B and C are disjoint, so the cardinality of their union is the sum of their cardinalities.

c. 2. $C = \{20, 40, 60\}$, so removing the one common element with A (20) leaves $C \setminus A = \{40, 60\}$.

d. 2. $B \cup D = B$, and B has two elements.

e. 0. $C \times D = \varnothing$ since no pairs can be formed with right-hand element from $D = \varnothing$.

f. 9. There are 3 elements of C and 3 of A, so 9 possible pairs.

g. 0. $D \cap E = \varnothing$ since $D = \varnothing$.

h. 8. All elements of $\mathcal{P}(A)$ are given by either including or not including each element of A in a given subset. There are 3 elements for each of which we can independently make one of these two choices, so $|\mathcal{P}(A)| = 2^3 = 8$.

i. 1. Recall that $\mathcal{P}(\varnothing) = \{\varnothing\}$.

j. 2^{24}. $|\mathcal{P}(A)| = 8$ and $|C| = 3$, so $|\mathcal{P}(A) \times C| = 24$ by similar reasoning to part (f). Then the powerset of this set has 2^{24} elements via the process for forming subsets described in part (h).

Problem 2

Let A, B, and C be arbitrary sets of natural numbers.

Prove or disprove the following facts. Write clear, carefully structured proofs! To show a set equality, *use the set-element method.* Your proofs might not be *long*, but you should state every step of your argument.

a. $\{x : \mathbb{Z} \mid x < 1\} \cap \{x : \mathbb{Z} \mid x > -2\} \subseteq \{x : \mathbb{Z} \mid x^2 = x\}$
b. $\{t : \mathbb{Q} \mid t^2 - 5t + 6 = 0\} = \{3\}$
c. $(A \setminus B) \setminus C = A \setminus (B \cup C)$
d. $\mathcal{P}(\{1, 2\} \times \{3, 4\}) \subseteq \mathcal{P}(\{1, 2, 3, 4\})$

Solution:

a. This is false.

   In order for the subset relation to obtain, every element of the left-hand set must be an element of the right-hand set. Observe that -1 is an element of the LHS: $-1 \in \{x : \mathbb{Z} \mid x < 1\}$ because $-1 < 1$, and $-1 \in \{x : \mathbb{Z} \mid x > -2\}$ because $-1 > -2$, so by the definition of intersection, $-1 \in \{x : \mathbb{Z} \mid x < 1\} \cap \{x : \mathbb{Z} \mid x > -2\}$. However, -1 is not an element of the right-hand set because $(-1)^2 = 1 \neq -1$.

   So -1 is an element of the left-hand set but not the right-hand one; therefore, the LHS cannot be a subset of the RHS.

b. This is false.

   By extensionality, equality of two sets requires that each be a subset of the other. So, in particular, if these sets are equal, then the LHS is a subset of the RHS, so every element of the LHS is an element of the RHS. However, observe that $2 \in \{t : \mathbb{Q} \mid t^2 - 5t + 6 = 0\}$ because $4 - 10 + 6 = 0$, but $2 \notin \{3\}$ by observation. So by the same reasoning as in part (a), the LHS is not a subset of the RHS; accordingly, the LHS cannot equal the RHS.

c. This is true. We proceed by the set-element method: we show that each set is a subset of the other. Let $x : \mathbb{N}$ be arbitrary.

   $\subseteq$ Suppose $x \in (A \setminus B) \setminus C$. Our goal is to show $x \in A \setminus (B \cup C)$.
   By the definition of set difference, we have $x \in (A \setminus B)$ and $x \notin C$. By the definition of set difference again, the former implies $x \in A$ and $x \notin B$. Since $x \notin B$ and $x \notin C$, that is, x is not in *either* B or C, we have $x \notin (B \cup C)$ (this is De Morgan's Law: $x \notin B \land x \notin C$ is equivalent to

$\neg(x \in A \vee x \in B)$). So now we have that $x \in A$ and $x \notin B \cup C$, so by the definition of set difference, we have $x \in A \setminus (B \cup C)$, as desired.

$\supseteq$ Suppose $x \in A \setminus (B \cup C)$. Our goal is to show $x \in (A \setminus B) \setminus C$.

By the definition of set difference, we have $x \in A$ and $x \notin B \cup C$. The second implies that it is neither the case that x is in B nor that x is in C; symbolically, $\neg(x \in B \vee x \in C)$. By De Morgan's Law, this is equivalent to the claim that x is not in B and x is not in C (i.e., $x \notin B \wedge x \notin C$). So, in particular, we have $x \in A$ and $x \notin B$, so by the definition of set difference, we have $x \in A \setminus B$. Then, since we further have that $x \notin C$, we apply the definition of set difference again to conclude that $x \in (A \setminus B) \setminus C$, as desired.

d. This is false. As in parts (a) and (b), it suffices to exhibit an element of the left-hand set that is not an element of the right-hand set. We claim that $\{(1,3)\}$ is such a value.

First, we show that $\{(1,3)\} \in \mathcal{P}(\{1,2\} \times \{3,4\})$. By the definition of power set, we must show that $\{(1,3)\} \subseteq \{1,2\} \times \{3,4\}$. Since $1 \in \{1,2\}$ and $3 \in \{3,4\}$, it follows by the definition of the Cartesian product that this is the case.

Next, we show that $\{(1,3)\} \notin \mathcal{P}(\{1,2,3,4\})$. By the definition of power set, this is equivalent to showing $\{(1,3)\} \not\subseteq \{1,2,3,4\}$. But clearly $(1,3)$ is an element of the left-hand set but not of the right-hand set, since $(1,3)$ is a tuple but all of the elements of the right-hand set are natural numbers. Accordingly, the left-hand set is not a subset of the right-hand one.

So we have found a value that is an element of $\mathcal{P}(\{1,2\} \times \{3,4\})$ but not $\mathcal{P}(\{1,2,3,4\})$, and we therefore conclude that the former is not a subset of the latter.

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Homework03.lean` in the directory browser on the left of your screen in your Codespace. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, download the file to your computer (right-click on the file in the Codespace directory browser and click "Download"), and upload it to Gradescope.

Problem 4 (Mind Bender — *Extra Credit*)

Recall that we can create sets with *descriptions* using set-builder notation, like

$$A = \{x : \mathbb{R} \mid x^2 > 1\}.$$

We would read this as "A is the set of x in the real numbers *such that* $x^2 > 1$." The condition that $x^2 > 1$ is a condition we place on this set to define it. Another example could be

$$C = \{d : \mathit{Dinosaur} \mid d \text{ is a carnivore}\}.$$

to define C to be the set of all dinosaurs that eat meat. This is an example of a description that is in natural language. One might consider if *every* description is a valid description.

Let X be a set, and consider the description "X does not contain itself".

For example, let

$$C = \{d : \mathit{Dinosaur} \mid d \text{ is a carnivore}\}.$$

be the set of all meat-eaters. This set C does not contain itself ($C \notin C$) since the set of all meat-eaters is not a meat-eater.

However, let

$$U = \{X : \mathit{Set} \mid \emptyset \subseteq X\}$$

be the set of all sets that are supersets of the empty set. It's true that $\emptyset \subseteq U$, since every set is a superset of the empty set; so therefore $U \in U$.

Let's build a set with this specific description. Let S be the set that contains all sets that do not "contain themselves". That is, we define it to be

$$S = \{X : \mathit{Set} \mid X \notin X\}.$$

A set such as U *would not* be in S, as U contains itself. However, a set such as C *would* be in S, as C does not contain itself.

a. Show that the assumption that S is a member of S leads to a contradiction.

b. Show that the assumption that S is not a member of S leads to a contradiction.

c. What do these contradictions suggest about how we can or cannot define a set? This paradox is called *Russell's Paradox*. Are there any ways to resolve these contradictions in set theory? Do some research and cite at least one source.[1]

[1] This is an open ended question, and there is no right answer! You should demonstrate to us that you have an understanding of what is going on.

Solution:

Part A: Assume that S is a member of S. If S is a member of S, then by the definition of S, S does not contain itself. But if S is a member of S, then S *does* contain itself. We therefore have that, if S is a member of S, S is not a member of S. That is, $S \in S \to S \notin S$.

Part B: Assume that S is not a member of S. If S is not a member of S, then S should be in S, as S contains all sets that do not contain themselves. We therefore have that if S is not a member of S, S is a member of S. That is, $S \notin S \to S \in S$.

Part C: (Answers will vary) This contradiction is called Russell's Paradox. Russell's paradox shows that the unrestricted axiom of comprehension (the idea that any condition or property may be used to determine a set) needs restriction because it can lead to contradictions. Russell's paradox is also motivated by adoption of the vicious circle principle (no propositional function can be defined prior to specifying the function's scope of application).

A mathematician named Ernst Zermelo then adapted a new form of Set Theory where the previous axiom of there existing a set containing any elements that satisfy a given property was replaced by an axiom that says that for any combination of set and property, there exists a new set of elements that are in the original set and that satisfy the property. This new axiomatic foundation avoids Russell's Paradox, and defines Zermelo-Frankel Set Theory.

Source: `https://www.scientificamerican.com/article/what-is-russells-paradox/`