Homework 3

Due: Friday, February 23, 2024

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

Let $A = \{0, 10, 20\}$, $B = \{\text{T-rex}, \text{brachiosaurus}\}$, $C = \{k : \mathbb{Z} \mid k/2 \in \{10, 20, 30\}\}$, $D = \{\}$, and $E = \mathbb{N}^+$.

Find the cardinalities of the following sets. Justify your answers.

a. $A \cap C$

b. $B \cup C$

c. $C \setminus A$

d. $B \cup D$

e. $C \times D$

f. $C \times A$

g. $D \cap E$

h. $\mathcal{P}(A)$

i. $\mathcal{P}(D)$

j. $\mathcal{P}(\mathcal{P}(A) \times C)$

Problem 2

Let A, B, and C be arbitrary sets of natural numbers.

Prove or disprove the following facts. Write clear, carefully structured proofs! To show a set equality, *use the set-element method.* Your proofs might not be *long*, but you should state every step of your argument.

a. $\{x : \mathbb{Z} \mid x < 1\} \cap \{x : \mathbb{Z} \mid x > -2\} \subseteq \{x : \mathbb{Z} \mid x^2 = x\}$

b. $\{t : \mathbb{Q} \mid t^2 - 5t + 6 = 0\} = \{3\}$

c. $(A \setminus B) \setminus C = A \setminus (B \cup C)$

d. $\mathcal{P}(\{1, 2\} \times \{3, 4\}) \subseteq \mathcal{P}(\{1, 2, 3, 4\})$

Problem 3

This problem is a Lean question!

This homework question can be found by navigating to `BrownCs22/Homework/Homework03.lean` in the directory browser on the left of your screen in your Codespace. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, download the file to your computer (right-click on the file in the Codespace directory browser and click “Download”), and upload it to Gradescope.

Problem 4 (Mind Bender — *Extra Credit*)

Recall that we can create sets with *descriptions* using set-builder notation, like

$$A = \{x : \mathbb{R} \mid x^2 > 1\}.$$

We would read this as "A is the set of x in the real numbers *such that* $x^2 > 1$." The condition that $x^2 > 1$ is a condition we place on this set to define it. Another example could be

$$C = \{d : \mathit{Dinosaur} \mid d \text{ is a carnivore}\}.$$

to define C to be the set of all dinosaurs that eat meat. This is an example of a description that is in natural language. One might consider if *every* description is a valid description.

Let X be a set, and consider the description "X does not contain itself".

For example, let

$$C = \{d : \mathit{Dinosaur} \mid d \text{ is a carnivore}\}.$$

be the set of all meat-eaters. This set C does not contain itself ($C \notin C$) since the set of all meat-eaters is not a meat-eater.

However, let

$$U = \{X : \mathit{Set} \mid \emptyset \subseteq X\}$$

be the set of all sets that are supersets of the empty set. It's true that $\emptyset \subseteq U$, since every set is a superset of the empty set; so therefore $U \in U$.

Let's build a set with this specific description. Let S be the set that contains all sets that do not "contain themselves". That is, we define it to be

$$S = \{X : \mathit{Set} \mid X \notin X\}.$$

A set such as U *would not* be in S, as U contains itself. However, a set such as C *would* be in S, as C does not contain itself.

a. Show that the assumption that S is a member of S leads to a contradiction.

b. Show that the assumption that S is not a member of S leads to a contradiction.

c. What do these contradictions suggest about how we can or cannot define a set? This paradox is called *Russell's Paradox*. Are there any ways to resolve these contradictions in set theory? Do some research and cite at least one source.[1]

[1] This is an open ended question, and there is no right answer! You should demonstrate to us that you have an understanding of what is going on.