Homework 2

Due: Wednesday, February 12, 2025

All homeworks are due at 11:59 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problems marked with a \ast are problems which may appear on the midterm or final with some modification.

Problem 1

Determine which of the following statements are true and justify your answer. Note the different domains of quantification for each statement. Below, $\mathbb{Z}_{\neq 0}$ is the nonzero integers and $\mathbb{R}_{\neq 0}$ is the nonzero real numbers.*

- a. $\forall x : \mathbb{R}_{\neq 0}, \exists y : \mathbb{R}_{\neq 0}, x \cdot y = 1.$
- b. $\forall x : \mathbb{Z}_{\neq 0}, \exists y : \mathbb{Z}_{\neq 0}, x \cdot y = 1.$
- c. $\forall x : \mathbb{Z}_{\neq 0}, \exists y : \mathbb{R}_{\neq 0}, x \cdot y = 1.$
- d. $\forall x : \mathbb{R}_{\neq 0}, \exists y : \mathbb{Z}_{\neq 0}, x \cdot y = 1.$

Problem 2

Welcome to the CS22 Saloon! We are about to open the saloon, but we need to make sure that everyone that visits, no matter the language, can understand what is going on! Translate the sentences below into formulas of first-order logic. Note: the sets listed here can be used as *domains* of quantification. That is, you could write $\forall x : D, \ldots$ to quantify over all townsfolk. You should not use any other domains of quantification.

- Sets:
 - D: The set of all townsfolk
- Predicates
 - B(x): "x is a bandit"
 - -C(x): "x is a cowboy"
 - F(x, y): "x and y are friends"
 - $<, \leq, \geq$, and > have their familiar meanings when applied to numbers.
- Functions
 - -w(x): the amount of food x eats per day
 - -h(x): the height of x (in feet)
- Constants
 - -s: Sawyer, the town Sheriff
 - -j: Jesse, the most dangerous outlaw in town
- a. There is a bandit that is friends with a cowboy.*
- b. There is no bandit that is taller than Sawyer.*
- c. No bandit eats less food than every cowboy. *
- d. Jesse is shorter than Sawyer, but Jesse is taller than every cowboy's friends.*

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Now consider the sentence "Every townsperson over 6 feet tall is friends with at least one bandit." This sentence translates to

$$\forall x: D, h(x) > 6 \to \exists y: D, B(y) \land F(x, y)$$

(Think about why!)

Suppose I wanted to prove this claim. I might begin by writing:

Fix an arbitrary townsperson d, and suppose d is over 6 feet tall.

- e. This sentence corresponds to two proof rules: which ones?
- f. What does the "proof state" look like after this start? Describe it like the Lean infoview: what is the goal (or goals), and what is the context of each goal? *
- g. If you were going to continue this proof, what proof rule would you use next?

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Problem 3

a. Consider the following formula in propositional logic:

$$((a \lor b) \land (\neg a \lor c)) \to (b \lor c)$$

- i. Is this formula *satisfiable*? Is it *valid*? (You do not need to justify your answers.)
- ii. Suppose we replace the formula above with one using very many propositional atoms, and insist that it is *satisfiable*. We (your humble instructors) are human and you shouldn't trust us unconditionally. What kind of evidence would convince you that it is, in fact, satisfiable? Explain in one or two sentences.
- iii. We're strengthening our claim: the formula is *valid*. Again, what kind of evidence would convince you of this?
- b. Now we switch to first order logic. For some domain D, let P(x) be a unary predicate symbol. We do *not* specify what this predicate means.
 - i. The notion of "validity" in first order logic is trickier than in propositional logic. Can you think of a first order formula in this language, using at least one quantifier, that is true no matter what D and P mean?
 - ii. Suppose we presented you with a complicated formula in this language and claimed it was "valid" in this sense. How could we convince you of that?

Problem 4

This problem is a Lean question!

This homework question can be found by navigating to BrownCs22/Homework/Homework02.lean in the directory browser on the left of your screen in your Codespace. The comment at the top of that file provides more detailed instructions.

You will submit your solution to this problem separately from the rest of the assignment. Once you have solved the problem, download the file to your computer (right-click on the file in the Codespace directory browser and click "Download"), and upload it to Gradescope.



Problem 5 (Mind Bender — Extra Credit)

Back to propositional logic for a moment!

We introduce our own new connective, "lasso," written \circ (\circ in LAT_EX). Lasso follows these proof rules:

- 1. (o introduction left) To prove $p \circ q,$ it suffices to assume p and derive a contradiction.
- 2. (o introduction right) To prove $p \circ q$, it suffices to assume q and derive a contradiction.
- 3. (o elimination) If you know $p \circ q$, p, and q, then you know r for any proposition r.

We will ask you to prove some facts about lasso in this mindbender. You may write your proofs in informal language, or justify your claims with truth tables. We will be as picky as Lean when we read these justifications!

- a. Show that $p \circ p$ is logically equivalent to $\neg p$.
- b. Using part a., show that $(p \circ p) \circ (p \circ p)$ is logically equivalent to p.
- c. Show that $p \lor q$ is logically equivalent to $(p \circ p) \circ (q \circ q)$.
- d. Show that $p \wedge q$ is logically equivalent to $(p \circ q) \circ (p \circ q)$.
- e. Find a propositional formula that uses *only* the connective \circ that is equivalent to $p \to q$.
- f. Our familiar quantifiers \land , \lor , \rightarrow , etc. all have "intuitive" English meanings. What meaning does \circ seem to capture?