Contents

1 Induction

1.1 Template and Weak Induction

Induction is a proof method for which we can assume some n case, and prove that every $n+1$ case holds. If we can prove that the $n+1$ case holds, we can confirm that our original claim holds for all values of n in the desired domain..

Idea: If you are stuck on an induction problem on the exam, start by writing out the inductive hypothesis and the structure of the proof. You will receive partial credit for this and it will also help you think of how to proceed.

Idea: Often the inductive step is a direct proof using the inductive hypothesis. This is not always the case; sometimes you might have to use ***proof by cases*** *or even* ***contradiction****.*

We will first provide a review of the template for an inductive proof and provide an example.

Example

For example, say we are trying to prove that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ is true for all $n \in \mathbb{N}$.

1. Define the predicate $P(n)$.

   Let $P(n)$ be the predicate that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

2. Show that the base case is true.

   We will first show $P(0)$ is true. $\sum_{i=0}^{0} i = 0$ and $\frac{0(0+1)}{2} = 0$ so they are equal as needed.

3. Assume the inductive hypothesis is true. If you are using stardard induction then you will assume $P(k)$ is true for some integer k. If you are using strong induction then you will assume $P(i)$ is true for all $i \leq k$. Either way, you should specify that k is some integer greater than or equal to your greatest base case.

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$.

4. Show that $P(k+1)$ is true given the inductive hypothesis.

   We will now show that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

   We know that $\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$.

   By our inductive hypothesis $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Therefore

$$\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$$
$$= \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

as needed. □

5. Conclude the proof.

   Therefore, as $P(0)$ is true and $P(k)$ implies $P(k+1)$ for all $k \in \mathbb{Z}$, $k \geq 0$, $P(n)$ is true for all nonnegative integers n.

Practice Problem(s)

1. Show that $6 \mid (n^3 - n)$ for all $n \in \mathbb{N}$.

2. Recall that the Fibonacci numbers are defined by $f_0 = 0, f_1 = f_2 = 1$ and the recursive relation $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. Challenging exercise:

   Show that f_n and f_{n+1} are 'relatively prime' for all $n \geq 1$. That is, they share no factor in common other than the number 1.

3. Use induction to show that $12^n + 2(5^{n-1})$ is divisible by 7 for all $n \geq 1$.

Solution(s)

1. Let $P(n)$ be the predicate that 6 divides $(n^3 - n)$ for some natural number n. Then, the base case is when $n = 0$, so $n^3 - n = 0^3 - 0 = 0$ is divisible by 6 meaning the base case is proven.

   Assume that $P(k)$ is true for some arbitrary integer $k \geq 0$. Then $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + (3k^2 + 3k) = (k^3 - k) + 3k(k+1)$.

   The left term is divisible by 6 according to the inductive hypothesis. $n(n+1)$ is always an even quantity, so let $n(n+1) = 2a$ for some integer a. Then, we get $(k^3 - k) + 3(2a) = (k^3 - k) + 6a$ is divisible by 6. Thus, since the base case is true and $\forall k \in \mathbb{N}, P(k) \implies P(k+1)$, $P(n)$ is true for all natural numbers.

2. Let $P(n)$ be the predicate that $f(n)$ and $f(n+1)$ are relatively prime. Then for the base case $n = 1$, we see that $f_1 = 1 = f_2$ have only the common factor 1. Thus, $P(1)$ holds.

Assume the inductive hypothesis. Then $f_{k+1} = f_k + f_{k-1}$. By the inductive hypothesis, we know that f_k and f_{k-1} are relatively prime.

We can factor each term into primes and write $f_{k+1} = f_k + f_{k-1} = p_1 p_2 ... p_a + p'_1 p'_2 ... p'_b$. (Since f_k and f_{k+1} are relatively prime, their factorizations should contain no common primes!) Recall that we want to show that f_{k+1} and f_k are relatively prime. That is, we want to prove that $\gcd(f_{k+1}, f_k) = 1$. If $\gcd(f_{k+1}, f_k) \neq 1$, then $\gcd(f_{k+1}, f_k) = p_i$ for $1 \leq i \leq a$. This leads to a contradiction, however, since the right hand term is not divisible by any of the primes in the factorization of f_k, so $\gcd(f_{k+1}, f_k) = 1$ meaning f_{k+1} and f_k are relatively prime.

Since we have shown that $P(1)$ holds and that $\forall k \geq 1, P(k) \implies P(k+1)$, the statement has been proved.

3. Let $P(n)$ be the predicate that $12^n + 2(5^{n-1})$ is divisible by 7. Then the base case is that $12 + 2 = 14$ is divisible by 7, so $P(1)$ holds.
   Now assume that $P(k)$ holds for an arbitrary $k \geq 1$. We need to prove that $P(k) \implies P(k+1)$.
   $P(k+1)$ would mean that $12^{k+1} + 2(5^k)$ is divisible by 7. Simplifying, we get $12(12^k) + 10(5^{k-1})$. We can write this as $12(12^k + 2(5^{k-1})) - 14(5^{k-1})$. From the inductive hypothesis, we know that $12^k + 2(5^{k-1})$ is divisible by 7, so $12(12^k + 2(5^{k-1}))$ is too. The second term, $-14(5^{k-1})$ is also divisible by 7 because 14 is divisible by 7. Since $P(k) \implies P(k+1)$, the theorem is proven.

More generally, our induction doesn't have to start from 0 but can start from any $n_0 \geq 0$. In particular, induction still works if our goal is to show $\forall n \geq n_0 \;\; p(n)$ where here this quantification is over all natural numbers at least n_0.

1.2 Strong Induction

The difference in approach between weak and strong induction comes in the induction hypothesis! In weak induction, we only assume that the predicate holds for some arbitrary step k, while in strong induction, we assume that the predicate holds at all steps from the base case to some arbitrary step k. Your inductive step may differ depending on whether you approach a problem using weak or strong induction, but they are equivalent!

Why would we need to do that? Sometimes, you can't just rely on the fact that $P(k)$ is true. Maybe you also need $P(k-1)$ to be true, or perhaps also $P(k-2)$, or even $P(k/2)$. While writing out your inductive step, if you realize that $P(k)$ isn't enough to prove $P(k+1)$, odds are, you need strong induction.

Practice Problem(s)

1. Define the sequence S as follows: $S_1 = 1$, $S_2 = 3$, $S_n = S_{n-1} * S_{n-2}$ for integers $n \geq 2$. Prove that S_n is odd for all positive integers n.

2. Prove that every positive integer greater than one can be factored as a product of

primes, using strong induction.

Solution(s)

1. Let $P(n)$ be the predicate that S_n is an odd number.
   Base cases:

   - $P(1)$ holds because $S_1 = 1$ is odd.
   - $P(2)$ holds because $S_2 = 3$ is odd.

   Inductive hypothesis: Assume $P(i)$ holds for all i such that $1 \leq i \leq k$, where k is an arbitrary integer such that $k > 2$.
   We need to prove that $P(k+1)$ holds, i.e. that $S_{k+1} = S_k * S_{k-1}$ is odd. From our inductive hypothesis, we know that $P(k)$ and $P(k-1)$ hold, so S_k and S_{k-1} are odd. The product of odd numbers is odd, so S_{k+1} is odd too, proving $P(k+1)$. Since $P(k-1)$ and $P(k)$ together imply $P(k+1)$, we have proven the theorem.

2. Let $P(n)$ be the predicate that n can be factored as a product of primes.
   Base case: $P(2)$ holds because 2 is a product of itself and it's prime.
   Inductive hypothesis: Assume that $P(i)$ holds for all $2 \leq i \leq k$, where $k \geq 2$.
   Inductive step: We need to prove that $P(k+1)$ holds. We have two cases:

   - If $k+1$ is prime, it is a product of itself, so $P(k+1)$ holds.
   - If $k+1$ is composite, it can be represented as a product of two numbers, a and b, where $2 \leq a, b < k+1$. According to our induction hypothesis, both a and b can be represented as a product of primes. When we multiply a and b in their factored form, we will obtain $k+1$ as a product of primes. Since our inductive hypothesis implies $P(k+1)$ in all cases, the theorem is proven.

2 Number Theory

Number theory is the study of the integers. For this section, all numbers are integers.

2.1 Definitions

Definition 1: We say that a *divides* b, denoted $a \mid b$, when $b = ka$ for some $k \in \mathbb{Z}$.

Definition 2: We say that a is *congruent* to b mod m, denoted $a \equiv b \pmod{m}$, if $m \mid (b - a)$. Another way to say this is that $a = b + km$ for some $k \in \mathbb{Z}$. Yet another way to say this: a and b have the same remainder upon division by m. Take a moment to convince yourself that these statements are equivalent.

2.2 Properties of Congruence Relations:

For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{m}$,

1. $a + c \equiv b + c \pmod{m}$ for any $c \in \mathbb{Z}$
2. $ac \equiv bc \pmod{m}$ for any $c \in \mathbb{Z}$
3. $a^n \equiv b^n \pmod{m}$ for $n \in \mathbb{Z}^+$

If we also have $c \equiv d \pmod{m}$,

1. $a + c \equiv b + d \pmod{m}$
2. $ac \equiv bd \pmod{m}$

2.3 GCD

The greatest common denominator of a and b is the largest positive integer which divides both a and b. To find the gcd of two numbers, we can run the Euclidean algorithm.

Theorem 1: For any $a, b \in \mathbb{Z}$ there exists $u, v \in \mathbb{Z}$ such that $au + bv = \gcd(a, b)$. In words, we say that a and b can be written as a linear combination of their gcd.

Theorem 2: An integer is a linear combination of a and b if and only if it is a multiple of their gcd.

Practice Problem(s)

1. Calculate $\gcd(1147, 899)$ using the Euclidean algorithm.
2. Calculate $\gcd(556, 148)$ using the Euclidean algorithm.
3. A bug is standing on a grid and can take four possible kinds of steps: $(9, 2)$, $(-12, 3)$, $(3, -6)$, $(-9, -12)$. Prove that if the bug starts at $(0, 0)$ then it can never reach $(1, 1)$.

Solution(s)

1. We use Euclid's Algorithm.

$$1147 = 899 \times 1 + 248$$

$$899 = 248 \times 3 + 155$$
$$248 = 155 \times 1 + 93$$
$$155 = 93 \times 1 + 62$$
$$93 = 62 \times 1 + 31$$
$$62 = 31 \times 2$$

So $\gcd(1147, 899) = 31$.

2. We use the Euclid's Algorithm.

$$556 = 148 \times 3 + 112$$

$$148 = 112 \times 1 + 36$$
$$112 = 36 \times 3 + 4$$
$$36 = 4 \times 9$$

So $\gcd(556, 148) = 4$.

3. Consider the x-coordinates of the bug's possible movements. The gcd of these is

$$\gcd(9, -12, 3, -9) = 3$$

Thus, if we let x be the bug's x-coordinate at any time, $x \equiv 0 \pmod 3$. Since $1 \equiv 1 \pmod 3$, our x-coordinate will never be 1, so the bug will never be at $(1, 1)$.

2.4 Multiplicative Inverse

Consider the particular congruence

$$ax \equiv 1 \pmod m.$$

If this equation has a solution, then we know we can find some integer x which, when multiplied by a, yields $1 \pmod m$. We define this integer to be the *multiplicative inverse* of $a \pmod m$, and we denote it a^{-1}. If a multiplicative inverse exists $\pmod m$, then when working $\pmod m$, we can "divide" by a—that is, we can multiply two sides of a congruence by a^{-1}, cancelling a from both sides.

When does a multiplicative inverse exist? According to the above Theorem 2: a^{-1} exists if and only if $\gcd(a, m)$ divides 1 (which is c in this particular congruence.) Thus, a^{-1} exists $\pmod m$ if and only if $\gcd(a, m) = 1$, that is, if and only if a and m are relatively prime. How do we find the multiplicative inverse? We can run the Euclidean algorithm and then backtrack to obtain the multiplicative inverse (gcdcombo).

2.5 Fermat's little Theorem

If p is prime and does not divide $a \in \mathbb{Z}$ then

$$a^{p-1} \equiv 1 \pmod{p}.$$

This means a^{p-2} is a multiplicative inverse for a mod p.

2.6 Euler's Totient Function

The totient function of n is a count of how many positive integers less than or equal to n are relatively prime to it. For any prime p, $\phi(p) = p - 1$. If m and a are relatively prime, then

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

This means $a^{\phi(m)-1}$ is a multiplicative inverse for a mod m. Fermat's little theorem is just a special case of this rule.

Practice Problem(s)

1. Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a + c \equiv b + d \pmod{m}$.

2. Use Fermat's Little Theorem or Euler's Theorem to find a multiplicative inverse of these numbers mod n in $[0, n)$. If the inverse does not exist, show why it does not exist.

   a) $14^2 \pmod{5}$

   b) $1452 \pmod{9}$

   c) $4^6 \pmod{15}$

Solution(s)

1. Assume $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Then $\exists x, y, k \in \mathbb{Z}$ such that

$$m \cdot x + k = a$$
$$m \cdot y + k = b$$

And $\exists u, v, w \in \mathbb{Z}$ such that

$$m \cdot u + w = c$$
$$m \cdot v + w = d$$

So

$$a + c = m \cdot (x + u) + k + w$$
$$b + d = m \cdot (y + v) + k + w$$

Thus,

$$a + c \equiv b + d \pmod{m}$$

2. a) According to Fermat's little theorem, $14^4 \equiv 1 \pmod 5$. Therefore, $14^2 * 14^2 \equiv 1 \pmod 5$. By the definition of the multiplicative inverse, $14^2 = 196$ is a multiplicative inverse for itself. $196 \equiv 1 \pmod 5$, so the multiplicative inverse for 14^2 is 1.

   b) The inverse does not exist. This is due to the fact that $\gcd(9, 1452) = 3 \neq 1$, so 9 and 1452 are not relatively prime and 1452 does not have a multiplicative inverse $\pmod 9$.

   c) We use Euler's Theorem for this. Powering a number does not change the factors, and since 15 has none of the factors of 4, we know 15 and 4^6 are relatively prime. Thus, we can use Euler's Theorem. $\phi(15) = 8$, so we know $4^8 \equiv 1 \pmod{15}$. This is the same as writing $4^2 * 4^6 \equiv 1 \pmod{15}$. Therefore, $16 \pmod{15}$ is a multiplicative inverse for 4^6, so 1 is the multiplicative inverse.

3 Counting

3.1 Product Rule and Permutations

The **product rule** states that for finite sets $S_1, ..., S_n$, $|S_1 \times ... \times S_n| = |S_1| * ... * |S_n|$. This can be useful in representing how many ways we could make a series of n independent choices. If we know how many options we have for each choice, we can find the number of ways we could make all of the choices by multiplying all the numbers of options together.

If the choices are instead dependent on each other, so what we choose from S_1 affects what we can choose from S_2 but not the *number* of things we could choose from S_2, we can use the **generalized product rule**. The generalized product rule tells us that if we are making a sequence of length k and we have $n_1, ..., n_k$ options for each position, then there are $n_1 * ... * n_k$ total sequences we can form.

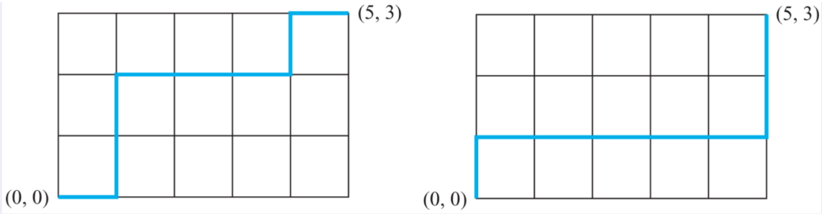
A **permutation** of a set A is an ordered list of the elements of A. The number of permutations of n elements is $n!$, which we can prove with the generalized product rule.

Example

The number of permutations of n elements is $n!$. This follows by the generalized product rule because we can break down constructing a sequence into choosing the first element for which there are n choices, then choosing the second element for which there are $n-1$ choices and so on for $n \cdot (n-1) \cdot (n-2) \cdot \ldots = n!$.

Practice Problem(s)

1. How many strings of eight English letters are there (considering y as a consonant)
   a) that contain no vowels, if letters can be repeated?
   b) that contain no vowels, if letters cannot be repeated?
   c) that start with a vowel, if letters can be repeated?
   d) that start with a vowel, if letters cannot be repeated?
   e) that contain at least one vowel, if letters can be repeated?
   f) that contain exactly one vowel, if letters can be repeated?
   g) that start with X and contain at least one vowel, if letters can be repeated?
   h) that start and end with X and contain at least one vowel, if letters can be repeated?
2. Count the number of paths in the xy plane between the origin $(0, 0)$ and point $(5, 3)$. Each path consists of a series of steps, where each step is a move one unit to the right or a move one unit upward. Two such paths are illustrated below:



3. (This was a scrapped homework problem that one of the TAs thought was fun - an equivalent exam question would be one part of this problem)

   Contrary to popular belief, the dinosaurs invented cards. Consider a standard deck of 52 cards. The dinosaurs played a game where each dino received 4 cards. For this problem, the order of the cards dealt does not matter, but the suits do.

   a) How many combinations of four card hands would a dino have **exactly** three of the same number (three of a kind)?

   b) How many combinations of four card hands would a dino have two pairs of different numbered cards (two-pair)?

   c) The dinos got bored and decided to invent the joker, a 53rd card that acts as a wild card, and can represent any number, but not a suit. (Ex: the joker can be a 2, but not the 2 of spades) **Assume that if the joker can be used to make a two-pair, it will.** Additionally, assume this is the only way the joker can be used, and otherwise it is just its own separate card with no suit.

      i) Repeat a) with the joker and the above assumption

      ii) Repeat b) with the joker and the above assumption

   d) After a million years, the dinos got bored again. **Now, assume that if the joker can be used to make three of a kind, it will.**

      i) Repeat a) with the joker and the above assumption

      ii) Repeat b) with the joker and the above assumption

   e) The dinos wanted to determine a winner for their game, and wanted the hand that was rarer, or had less combinations, to win. Lets say Dan has two pairs, and Brendan has three of a kind. List the winners for the three scenarios (a & b, c, d)

Solution(s)

1. We consider 5 vowels and 21 consonants.

   a) If letters can be repeated, we are making a sequence of length $k = 8$ with $n = 21$ options for each letter. Thus, we calculate 21^8 total strings.

b) If letters cannot be repeated, we are ordering a sequence of length $k = 8$ choosing from $n = 21$ values. Thus, we calculate the quotient of 21!, the number of ways to order the entire group, and $(21 - 8)! = 13!$, the number of ways to order the letters not chosen, or $\frac{21!}{13!}$ strings of distinct consonants.

c) If letters can be repeated and the first letter must be a vowel, we are making a sequence of length $k = 8$ with $n_1 = 5$ options for the first letter, and $n = 26$ options for each other letter. Thus, we calculate $5 * 26^7$ total strings.

d) We have 5 options for the first vowel, and then we have 25 options for the next letter because we can't duplicate the first letter, and so on. 5*25*24*...*19 total strings.

e) We count the total amount of strings possible, and then subtract the strings that contain only consonants. What's remaining will be the number of strings that are not made up of only consonants, which is the same as counting the number of strings that have at least one vowel. $26^8 - 21^8$ total strings.
WHY IT'S NOT $5 * 8 * 26^7$: this overcounts because some strings contain more than one vowel, and each string with multiple vowels would be counted once for each position chosen as "the vowel", so those strings are counted multiple times. The subtraction method $26^8 - 21^8$ avoids this by counting each string exactly once.

f) We have 5 options for the vowel and 8 possible spots to put it. The rest of the letters cannot be vowels, so we have 21 options for those. $5 * 8 * 21^7$ total strings.

g) If we start with X, we have 7 remaining spots, and one of those letters must be a vowel. We use the same strategy as part e to obtain $26^7 - 21^7$ total strings.

h) If we start and end with X, we have 6 remaining spots, and one of those letters must be a vowel. We use the same strategy as part e to obtain $26^6 - 21^6$ total strings.

2. In order to connect the two points, the path must contain exactly three steps up and five steps to the right, in any order. There are $\binom{8}{3} = \frac{8!}{5!3!}$ ways to order these eight steps, as we calculate the number of ways to order 8 distinct steps as a permutation, and then divide the ways to order 5 distinct steps and the ways to order 3 distinct steps, as steps in any one direction are non-distinct from each other.

3. a) There are $13 * 4$ possible three-of-a-kind sets, as there are 13 possible numerical values for the cards, and 4 possible suits that could be left out of each triple. The fourth card in each hand could be any of the other 12 numbers in any of the 4 suits. In total, that means that there are $13 * 4 * 12 * 4 = 2496$ four card hands where a dino would have exactly three of the same number.

   b) There are $\frac{52*3}{2} = 78$ possible un-ordered pairs of cards, as counting each of 52 cards can be paired with each of 3 other cards with the same value includes each pair exactly twice. An alternate argument yields that each of the 13 different card values has $\binom{4}{2}$ possible pairs that can be formed, so $13 * \binom{4}{2}$ also gives 78. A dino's two-pair hand could contain any of the 78 pairs coupled with any of the 72 other different-numbered pairs, with order irrelevant. Thus, we calculate

$\frac{78*72}{2} = 2808$ four card hands where a dino would have two pairs of different numbered cards.

c) i) There are still $13 * 4$ possible three-of-a-kind sets, as there are 13 possible numerical values for the cards, and 4 possible suits that could be left out of each triple. The fourth card in each hand could be any of the other 12 numbers in any of the 4 suits, or the joker. In total, that means that there are $13 * 4 * (12 * 4 + 1) = 2548$ four card hands where a dino would have exactly three of the same number.

ii) From part b, we know there are 2808 ways without using the joker. If we include the joker, there are $\frac{13 \cdot 4 \cdot 12 \cdot \binom{4}{2}}{2} = 1872$ additional ways. Therefore, there are $2808 + 1872 = 4680$ total ways.

d) i) From part a, we know there are 2496 ways without using the joker. If we include the joker, there are an additional $13 \cdot \binom{4}{2} \cdot 12 \cdot 4 = 3744$ ways where the joker is a part of the three-of-a-kind, and $13 * 4 = 52$ ways where the joker is the fourth card in a three-of-a-kind hand. Therefore, there are $2496 + 3744 + 52 = 6292$ total ways.

ii) We observe that any joker could make one of the two pairs into a 3 of a kind. Therefore, we can not include any joker in the four card hand, and the answer is the same as part b, 2808.

e)
- Brendan would win, as there are fewer ways to form three-of-a-kind than a two-pair according to the rules used in parts a & b.
- Brendan would win, as there are still fewer ways to form three-of-a-kind than a two-pair according to the rules used in part c.
- Dan would win, as there are now fewer ways to form two-pair than a three-of-a-kind according to the rules used in part d.

3.2 Binomial Coefficients and Theorem

The **binomial coefficient**, also called n *choose* k, is defined to be

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

for $n \geq k$ and $n, k \in \mathbb{Z}^+$.

The binomial coefficient $\binom{n}{k}$ counts the number of ways to choose k objects from n objects. Equivalently, its counts the number of subsets of size k of a set of size n.

Binomial Theorem: The coefficients of the terms in the polynomial $(x+y)^n$ are binomial coefficients, i.e.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Practice Problem(s)

1. What is the hundreds-place digit of 11^{2024}?

2. Find the middle term(s) of $(\frac{p}{x} + \frac{x}{p})^9$

Solution(s)

1. We can write 11^{2024} as $(10+1)^{2024}$. We need to find the terms that contain 1, 10, and 100.
$$\binom{2024}{0} * 10^0 * 1^{2024} = 1$$
$$\binom{2024}{1} * 10^1 * 1^{2023} = 20240$$
$$\binom{2024}{2} * 10^2 * 1^{2022} = 204727600$$
The hundreds digit will be 6+2=8.

2. The expansion will have 10 terms. This makes the middle terms the 5th and 6th. Let's find them:
5th term:
$$\binom{9}{4}(\frac{p}{x})^4(\frac{x}{p})^5 = 126\frac{x}{p}$$
6th term:
$$\binom{9}{4}(\frac{p}{x})^5(\frac{x}{p})^4 = 126\frac{p}{x}$$

3.3 Counting Arguments

A **counting argument** shows that the LHS (lefthand side) and the RHS (righthand side) of some equation count the same thing. Instead of using algebraic manipulation, we explain why both sides ultimately count the elements of some set, just in different ways.

Importantly, if a question asks you to use a counting argument, you cannot use the definition of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ or other algebraic arguments.

For instance, consider the following identity.
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Let S be a set with n elements. The LHS counts the number of ways to form a subset of S size k. Let x be some element of S. Each subset of S of size k either includes x or does not include

x. If the subset includes x, then we need to pick $k-1$ other elements for the subset from the remaining $n-1$ elements, which we can do in $\binom{n-1}{k-1}$ ways. If the subset does not include x, then we still need to pick all k elements, and can do so from the remaining $n-1$ elements since we can't pick x, which we can do in $\binom{n-1}{k}$ ways. So, adding these together to get the RHS, this also counts the number of subsets of S of size k.

Practice Problem(s)

1. Prove via counting argument that $1+2+\cdots+n = \binom{n+1}{2}$.

2. Prove via counting argument that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

Solution(s)

1. Let S be a set with $n+1$ elements. The RHS counts the number of ways to form a subset of S of size 2. Let x_i be the ith element of S. If x_i can be distinctly paired with any of the $n+1-i$ elements positioned after x_i in S, there are a total of $n + ... + 2+1$ distinct pairs contained in S. Thus, the LHS also counts the number of ways to form a subset of S of size 2.

2. Let S be a set with $2n$ elements. The LHS counts the number of ways to choose 2 elements from S.

   Then, we count the RHS. Suppose we divide S into two equally sized subsets S_1 and S_2, each with n elements. Then, to count the number of ways to choose 2 elements from S, we can take the sum of the number of ways to choose 2 elements from S_1, the number of ways to choose 2 elements from S_2, and the number of ways to choose 1 element each from S_1 and S_2. There are $\binom{n}{2}$ ways each to choose 2 elements from S_1 and S_2. To count the number of ways to choose 1 element each from S_1 and S_2, we use the product rule. There are n possible elements we can choose from S_1 and n possible elements we can choose from S_2, meaning there are n^2 ways we can choose 1 element from each of S_1 and S_2. Therefore, there are $\binom{n}{2} + \binom{n}{2} + n^2 = 2\binom{n}{2} + n^2$ total ways to choose 2 elements from S, and we can conclude the LHS and RHS are equal.

3.4 Inclusion/Exclusion Formula

The inclusion/exclusion formula provides a way of counting the size of a union of sets, and it is especially helpful if the sets overlap (and thus merely summing the sizes would result in over-counting.)

For two sets A and B, the inclusion/exclusion formula says that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

While the formula for three sets A, B, and C is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Going further, we can repeat this process for any number of sets, alternating between adding and subtracting the sizes of sets.

Practice Problem(s)

1. How many 6-digit numbers are even or are divisible by 5?
2. How many positive integers less than or equal to 1000 are multiples of 3, 5, or 7?
3. How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2, and
   a) $A_1 \cap A_2 = \emptyset$?
   b) $|A_1 \cap A_2| = 1$?
   c) $|A_1 \cap A_2| = 6$?
   d) $A_1 \subseteq A_2$?

Solution(s)

1. Let A represent the set of even 6-digit numbers.

   $$|A| = 9 * 10 * 10 * 10 * 10 * 5 = 450000.$$

   Let B represent the set of 6-digit numbers divisible by 5.

   $$|B| = 9 * 10 * 10 * 10 * 10 * 2 = 180000.$$

   Then, $A \cap B$ represents the set of 6-digit numbers that are both divisible by 5 and even, or are divisible by 10.

   $$|A \cap B| = 9 * 10 * 10 * 10 * 10 * 1 = 90000.$$

   The set of 6-digit numbers that are even or divisible by 5 can be written as $A \cup B$. By the inclusion/exclusion formula, we then know that

   $$|A \cup B| = |A| + |B| - |A \cap B| = 450000 + 180000 - 90000 = 540000,$$

   so there are 540,000 such numbers.

2. Let A represent the set of positive integers less than or equal to 1000 that are multiples of 3.
$$|A| = \lfloor \frac{1000}{3} \rfloor = 333.$$
Let B represent the the set of positive integers less than or equal to 1000 that are multiples of 5.
$$|B| = \lfloor \frac{1000}{5} \rfloor = 200.$$
Let C represent the the set of positive integers less than or equal to 1000 that are multiples of 7.
$$|C| = \lfloor \frac{1000}{7} \rfloor = 142.$$
Then, $A \cap B$ represents the set of positive integers less than or equal to 1000 that are multiples of 3 and 5, or multiples of 15.
$$|A \cap B| = \lfloor \frac{1000}{15} \rfloor = 66.$$
$A \cap C$ represents the set of positive integers less than or equal to 1000 that are multiples of 3 and 7, or multiples of 21.
$$|A \cap C| = \lfloor \frac{1000}{21} \rfloor = 47.$$
$B \cap C$ represents the set of positive integers less than or equal to 1000 that are multiples of 5 and 7, or multiples of 35.
$$|B \cap C| = \lfloor \frac{1000}{35} \rfloor = 28.$$
Finally, $A \cap B \cap C$ represents the set of positive integers less than or equal to 1000 that are multiples of 3, 5 and 7, or multiples of 105.
$$|A \cap B| = \lfloor \frac{1000}{105} \rfloor = 9.$$
The set of positive integers less than or equal to 1000 that are multiples of 3, 5, or 7 can be written as $A \cup B \cup C$. By the inclusion/exclusion formula, we then know that
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$
$$= 333 + 200 + 142 - 66 - 47 - 28 + 9 = 543,$$
so there are 543 such numbers.

3. By the inclusion/exclusion formula, we know that
$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$
Therefore,

a) $|A_1 \cup A_2| = 12 + 18 - 0 = 30$

b) $|A_1 \cup A_2| = 12 + 18 - 1 = 29$

c) $|A_1 \cup A_2| = 12 + 18 - 6 = 24$

d) $|A_1 \cup A_2| = |A_1| + 18 - |A_1| = 18$

3.5 Counting Donuts

The number of ways to distribute m identical objects among n distinct groups is

$$\binom{m+n-1}{n-1}.$$

Why is this? We can uniquely represent such a distribution with a 0/1 string of length $m+n-1$ that has exactly m 0's:

Let the m 0's represent the objects. The remaining $n-1$ bits in the string are 1's. Let all of the 0's to the left of the first 1 belong to group 1. Then, let all the 0's between the first 1 and the second 1 belong to the second group. Continue determining group membership in this fashion. Below is a diagram illustrating this. Note that, since 1's two and three are adjacent, nothing is in group 3.

$$\underbrace{0\ldots0}_{\text{group 1}}1\underbrace{0\ldots0}_{\text{group 2}}11\underbrace{0\ldots0}_{\text{group 4}}10\ldots01\ \underbrace{000}_{\text{group } n}$$

What's important to note is that (1) any distribution we choose can be represented with some length $m+n-1$ binary string, and (2) any such binary string represents a valid distribution of m identical objects into n distinct groups under this interpretation. In other words, the distributions and binary strings are in bijection with each other, meaning we can count one by counting the other.

We know how to count such binary strings: it is simply the number of ways you can choose $n-1$ of the bits to be 1's, leaving the other m bits to be 0's: $\binom{m+n-1}{n-1}$.

Practice Problem(s)

1. Solve each of the counting problems below with stars and bars.

   a) How many ways are there to select a handful of 6 jellybeans from a jar that contains 5 different flavors?

   b) How many ways can you distribute 5 identical lollipops to 6 kids?

   c) How many 6-letter words can you make using the 5 vowels in alphabetical order?

   d) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 6$?

2. Using the digits 2 through 9 (inclusive), find the number of 4-digit numbers such that:

   a. Digits cannot be repeated and must be written in increasing order. (Increasing means strictly increasing. For example, the digits of 134 are increasing, but the digits of 133 are not.)

b. Digits can be repeated and must be written in non-decreasing order. (Now the digits don't need to be strictly increasing; 133 has digits non-decreasing.)

3. How many integer solutions to $x_1 + x_2 + x_3 + x_4 = 25$ are there for which $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3$ and $x_4 \geq 4$?

4. Consider functions $f : \{1, 2, 3, 4, 5\} \to \{0, 1, 2, \cdots, 9\}$.

   a. How many of these functions are strictly increasing? Explain. (A function is strictly increasing provided if $a < b$, then $f(a) < f(b)$.)

   b. How many of the functions are non-decreasing? Explain. (A function is non-decreasing provided if $a < b$, then $f(a) \leq f(b)$.)

Solution(s)

1. a) We calculate the number of ways you can choose $5 - 1 = 4$ characters to be bars, separating the different flavors, leaving 6 characters to be stars that represent jellybeans: $\binom{10}{4} = \frac{10!}{4!6!} = 210$.

   b) We calculate the number of ways you can choose $6 - 1 = 5$ characters to be bars, separating the different kids, leaving 5 characters to be stars that represent lollipops: $\binom{10}{5} = \frac{10!}{5!5!} = 252$.

   c) We calculate the number of ways you can choose $5 - 1 = 4$ characters to be bars, separating the different vowels, leaving 6 characters to be stars that represent letters: $\binom{10}{4} = \frac{10!}{4!6!} = 210$.

   d) We calculate the number of ways you can choose $4 - 1 = 3$ characters to be bars, separating the different variables, leaving 6 characters to be stars that each represent the value 1: $\binom{9}{3} = \frac{9!}{3!6!} = 84$.

2. a) One way to solve this is by observing that simply picking 4 numbers from our set of numbers 2-9 will achieve every possible 4-digit number with no repeating digits. This is equivalent to what we're asking for in the problem because sorting those digits in increasing order will achieve every possible number with strictly increasing digits. Thus, the answer must be $\binom{8}{4}$.

   Another way to solve this problem is with stars and bars. First, notice that the lowest number we can get is 2345, and the highest is 6789. Second, notice that if we start from 2345, we can add at most 4 to any digit, and if we add $x \leq 4$ to some digit, we must add at least x to subsequent digits to attain a valid number (e.g. if we add 4 to the second digit in 2345, we must add 4 to both the third and fourth digits to attain 2789). The amount of ways we can do this adding is best represented by stars and bars. Let our stars represent additions by 1, and let the ith bar represent how many additions we will do to the ith digit. Consider this configuration, for example:

$$**||*|*|$$

Since the first bar and second bars follow two stars, we add 2 to both the first and second digits of 2345, obtaining 4545. Since the third bar follows 3 stars, and the fourth bar follows 4 stars, we add 3 and 4 to the third and fourth digits respectively, finally obtaining 4579. Also note that $||||****$ represents 2345, and $****||||$ represents 6789. Using this method, we see that we never add more than 4 to a single digit, and if we add $x \leq 4$ to the ith digit, we will add at least x to the $i+1$th digit. Thus, every number obtained is valid, and we count $\binom{4+4}{4} = \binom{8}{4}$ total valid numbers.

b) If digits can be repeated, we use 7 stars to represent the 8 digits between 2 and 9 – with the space after the stars representing the digit 9 – and four bars to represent the positions. Then, each digit's value corresponds to the value of the star directly following it. For example, the number 4559 would be written as

$$***|*||***|.$$

Thus, we calculate the number of ways you can choose 4 out of 11 characters to be bars: $\binom{11}{4} = \frac{11!}{4!7!} = 330$.

Alternatively, we can use $8-1=7$ bars to represent the 8 digits between 2 and 9. Then, we can use 4 stars to represent the value of each of the 4 digits. For example, the number 4559 corresponds to

$$||*|**||||*$$

This gives us the same total number of valid numbers, $\binom{4+8-1}{8-1} = \binom{11}{7} = \binom{11}{4}$.

3. We set the base values of the respective variables as $x_1 = 1, x_2 = 2, x_3 = 3$ and $x_4 = 4$. Then, we must increment the variables by a total of $25-1-2-3-4 = 15$ to make the equation valid. We can view this as a stars and bars problem, with three bars partitioning fifteen stars into four variables, with each star contributing $+1$ to the respective variable. Thus, we calculate the number of ways you can choose 3 out of 18 characters to be bars: $\binom{18}{3} = \frac{18!}{15!3!} = 816$.

4. See solution to problem #2 for explanation

   a) $\binom{10}{5}$

   b) $\binom{14}{5}$

3.6 Pigeonhole Principle

Pigeonhole Principle: If we put $k+1$ objects into k boxes, then some box has at least 2 objects. More generally, if we place n objects into k boxes, then some box must have at least $\lceil \frac{n}{k} \rceil$ objects.

Another way we can think about the Pigeonhole Principle is this. It tells us that if we have a function, $f : |X| \to |Y|$, such that the cardinality of X is n and the cardinality of Y is k, then there is some $y \in Y$ such that the number of $x \in X$ that map to y is greater than or equal to $\lceil \frac{n}{k} \rceil$.

Pigeonhole principle basically says that *some* box must have the average number of items per box (assume for the sake of contradiction that this were not the case—what would have to be true?) We get the ceiling function because we can't have fractional objects—objects must remain whole as they are placed into boxes.

Practice Problem(s)

1. What is the minimum number of times you must roll a six-sided dice before you can guarantee that 10 or more of the rolls resulted in the same number?

2. Prove that any set of seven distinct natural numbers contains a pair of numbers whose sum or difference is a multiple of 10.

3. A game comes with 40 six-sided dice (each numbered 1 to 6). Suppose you roll all 40 dice. Prove that there will be at least seven dice that land on the same number.

Solution(s)

1. After 54 rolls, in the worst case you could roll each number at most 9 times since $6 \times 9 = 54$. By the pigeon hole principle, the next roll (the 55th) forces some number to appear for the 10th time. Therefore, we need a minimum of 55 rolls.

2. We can create six distinct categories that encompass all natural numbers: (a) numbers that end in 0, (b) numbers that end in 1 or 9, (c) numbers that end in 2 or 8, (d) numbers that end in 3 or 7, (e) numbers that end in 4 or 6, and (f) numbers that end in 5. For a pair of numbers belonging to any of these categories, the pair's sum or difference is a multiple of 10. By the pigeonhole principle, if we group any $k + 1 = 7$ objects, or natural numbers, into these $k = 6$ boxes, or categories, some box will have at least two objects, or there will exist some pair of numbers whose sum or difference is a multiple of 10.

3. We can again think of this as placing $n = 40$ objects (dice) into $k = 6$ boxes (values), so some box must have at least $\lceil \frac{n}{k} \rceil = \lceil \frac{40}{6} \rceil = 7$ objects. Thus, there will be at least seven dice that land on the same number.