Strategies for Bijective Proofs

In this document, we walk through some strategies for proving that a mapping is a bijection.

Introduction

When thinking about what it takes for a mapping to be bijective, there are a few things to keep in mind:

1. Creating a mapping f between two sets A and B and showing that A and B are the same size does NOT prove that f is bijective. This is because there are lots of mappings between two sets of the same size that are not bijections! To convince yourself, consider two finite sets of the same size and come up with a mapping between them that isn't a bijection.

2. There are lots of injective mappings that are not surjective, and surjective mappings that are not injective. Therefore, when we want to show that a mapping is bijective, it is required of us to show both. Now, it is true that if two sets A and B are the same size, then an injective mapping between A and B is also a surjective mapping (and a surjective mapping is also an injective mapping). However, we are usually creating bijections to show that A and B are of the same size; that is, we don't know at the outset of the proof that A and B are of the same size! It is fallacious to argue something like:

a. We want to show that A and B are of the same size.

b. We have an injective mapping between A and B.

c. If A and B are of the same size, then this injective mapping is also surjective.

d. Therefore, this injective mapping is also surjective. [**The problem in the argument begins at this step: think about why.**]

e. So we can conclude that our mapping is bijective.

Therefore, in our proofs, we really do need to show that our mapping is both injective and surjective. We walk through the general strategies for how to do that below. We highly recommend taking a look at solutions to previous homework assignments to see the strategies in action!

Strategy for injectivity proofs

To prove that a mapping is injective, we want to start with a definition of injectivity and show that our mapping does in fact meet the definition. Note that there are a good number of ways to formulate injectivity, and each motivates a slightly different proof strategy. However, each definition is going to look something like *"if P, then Q"* (where P and Q are certain conditions), and we'll want to show that assuming P is the case, we must have that Q is the case too.

One definition of injectivity is: if we have an a and b in our domain such that $f(a) = f(b)$, then a must equal b. This motivates the following game plan for our proof:

1. We start with an a and b in our domain such that $f(a) = f(b)$. We make no assumptions about the equality of a and b.
2. However, we then show that because $f(a) = f(b)$, a is forced to be equal to b. That is, we show that given that our mapping output the same thing for a and b, we must have that a was equal to b in the first place. This cannot just be stated; the crux of the proof is convincing us that this is the case!

Another, equivalent definition of injectivity is that if a and b are elements in our domain such that $a \neq b$, then $f(a) \neq f(b)$. Note how this follows the format of *"if P, then Q"*. To prove that our mapping is injective starting with this definition, we want to do the same general strategy as before: start with P and show that Q must follow. Try to figure out the explicit proof strategy for this definition of injectivity, using the one above as a guide.

Strategy for surjectivity proofs

Like in injectivity proofs, we want to start with the definition of surjectivity and show that our mapping meets the definition. The most common formulation of surjectivity is the following:

Given an arbitrary b in our codomain, there exists an a in our domain such that $f(a) = b$. This motivates the following game plan for our proof:

1. Start with an arbitrary element b in the codomain.
2. Construct some a in the domain, keeping in mind that we want $f(a) = b$, and justify why a is in fact in the domain. If we don't justify why it's in the domain, we can't plug a into our mapping! That is, we could have constructed an a of some "type" that our mapping doesn't recognize.
3. We then show that our mapping applied to a results in the element of the codomain we started with, b. That is, we justify why $f(a) = b$. Note that it typically does not suffice to say something like "by construction, $f(a) = b$". Usually, we need to

walk through slowly and carefully why our construction process in fact led us to an a such that $f(a)$ is the element in the codomain we started with, b.

In general, having these game plans in your toolbox is important for two reasons. First and foremost, when we ask you to prove a mapping is bijective, then you'll have a game plan for yourself in solving the problem. Secondly, knowing this game plan means you can articulate to your readers the structure of what you're trying to do rather than just doing it, which is one of the necessary ingredients of a convincing argument.