# Bijective Proof Examples



**Problem 1.** Prove the existence of a bijection between 0/1 strings of length $n$ and the elements of $\mathcal{P}(S)$ where $|S| = n$

**Definition.** We define a function that maps every 0/1 string of length $n$ to each element of $\mathcal{P}(S)$. Let $f(a_1a_2\ldots a_n)$ be the subset of $S$ that contains the $i$th element of $S$ if $a_i = 1$ and does not contain the $i$th element if $a_i = 0$.

**Lemma.** *(injectivity) If $a_1a_2\ldots a_n \neq b_1b_2...b_n$, then $f(a_1a_2\ldots a_n) \neq f(b_1b_2\ldots b_n)$*

*Proof.* If $a_ia_2\ldots a_n \neq b_1b_2\ldots b_n$, then there is some $i$ such that $a_i \neq b_i$. Therefore, for this $i$, the $i$th element is either in $f(a_1a_2\ldots a_n)$ or in $f(b_1b_2\ldots b_n)$, but not both. Since the sets must differ by at least one element, they must be different sets. □

**Lemma.** *(surjectivity) For every subset of $S$, there exists some 0/1 string of length $n$ that is mapped to it.*

*Proof.* Let $A$ be a subset $\{n_1, n_2, \ldots, n_k\}$ with $k$ elements. Define $x$ to be the 0/1 string $x_1x_2\ldots x_n$, where $x_i = 1$ if the $i$th element is in $A$ and 0 otherwise. Then for every $A \subseteq$S, $\exists x$ such that $f(x) = A$. □

**Theorem.** *There exists a bijection from $\{0,1\}^n \to \mathcal{P}(S)$, where $|S| = n$.*

*Proof.* We have defined a function $f : \{0,1\}^n \to \mathcal{P}(S)$. Because $f$ is injective and surjective, it is bijective. □

**Problem 2.** Prove there exists a bijection between the natural numbers and the integers

**Definition.** Consider the following function that maps $\mathbb{N}$ to $\mathbb{Z}$:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{-(n+1)}{2} & \text{if n is odd} \end{cases}$$

**Lemma.** *(injectivity) If $a \neq b$, then $f(a) \neq f(b)$.*

*Proof.* Suppose that $a \neq b$ but $f(a) = f(b)$. Then $f(a)$ and $f(b)$ must have the same sign. Therefore, either $f(a) = \frac{a}{2}$ and $f(b) = \frac{b}{2}$ or $f(a) = \frac{-(a+1)}{2}$ and $f(b) = \frac{-(b+1)}{2}$. In both cases, solving for $a$ and $b$ gives $a = b$. □

**Lemma.** *(surjectivity) $\forall y \in \mathbb{Z}$, there exists some $x \in \mathbb{N}$ for which $f(x) = y$*

*Proof.* If $y$ is positive, then $f(2y) = y$ and $y$ has a "pre-image" equal to $2y$.

If $y$ is negative, then $f(-(2y+1)) = y$, and $y$ has a "pre-image" equal to $-(2y+1)$. □

**Theorem.** *There exists a bijection between $\mathbb{N}$, the natural numbers, and $\mathbb{Z}$, the integers.*

*Proof.* We have shown $f : \mathbb{N} \to \mathbb{Z}$ is injective and surjective. Therefore it is bijective. □

**Problem.** You want to buy 10 donuts from a shop that provides four flavors: french vanilla, garlic, java chip, and almond joy. Let $f$, $g$, $j$, and $a$ denote the number of each type of donut you buy. Prove the number of ways to buy 10 donuts from four flavors is equal to the number of 0/1 strings of length 13 that contain exactly three 1s.

*Remark.* We have two constraints. First, $f, g, j, a \geq 0$. Second, $f + g + j + a = 10$.

**Definition.** Consider the following function $h$ that maps length-13 0/1 strings with exactly three 1s to ways to buy 10 donuts from four flavors:

$$h(a_1a_2a_3\dots a_{13}) = (f, g, j, a)$$

where

$f$ is the number of 0s before the first 1

$g$ is the number of 0s between the first and second 1s

$j$ is the number of 0s between the second and third 1s

$a$ is the number of 0s after the third 1

**Lemma.** *(injectivity) If $a_1a_2\dots a_{13} \neq b_1b_2\dots b_{13}$, then $h(a_1a_2\dots a_{13}) \neq h(b_1b_2\dots b_{13})$*

*Proof.* We provide an informal proof by contradiction. Assume $a_1a_2...a_{13} \neq b_1b_2...b_{13}$ but $h(a_1a_2\dots a_{13}) = h(b_1b_2\dots b_{13})$. Let $(f_a, g_a, j_a, a_a) = h(a_1a_2\dots a_{13})$ and $(f_b, g_b, j_b, a_b) = h(b_1b_2\dots b_{13})$. By our assumption, $f_a = f_b$, $g_a = g_b$, $j_a = j_b$, and $a_a = a_b$. This necessarily implies that $a_1a_2\dots a_{13}$ and $b_1b_2\dots b_{13}$ have the same number of 0s before the first 1, the same number of 0s between the first and second 1s, the same number of 0s between the second and third 1s, and the same number of 0s after the third 1. This would mean that $a_1a_2\dots a_{13} = b_1b_2\dots b_{13}$ , contradicting our initial assumption that $a_1a_2\dots a_{13} \neq b_1b_2\dots b_{13}$. □

**Lemma.** *(surjectivity) For every $(f, g, j, a)$ there exists some length-13 0/1 string with exactly three 1s that maps to it.*

*Proof.* Assume you have a fixed $(f, g, j, a)$. Construct a 0/1 string as follows:

Write $f$ 0s, followed by a 1, then $g$ 0s, followed by a 1, then $j$ 0s, followed by a 1, then $a$ 0s. Then this string will be mapped to our fixed $(f, g, j, a)$ with the function we defined. □

**Theorem 3.** *The number of ways to buy 10 donuts from four flavors is equal to the number of 0/1 strings of length 13 that contain exactly three 1s.*

*Proof.* Because $h$ is injective and surjective, it is bijective.Because there exists a bijection between the number of ways to buy 10 donuts from four flavors and the number of 0/1 strings of length 13 that contain exactly three 1s, those numbers must be equal. □