CS22 Discrete Structures and Probability Practice Midterm Exam

With Solutions

Please read these directions carefully before you begin!

- This is based on an actual exam from CS22 last year.
- The practice midterm is not a replacement for reviewing this year's lectures, recitations, and homeworks.
- Modular arithmetic is not on this practice exam. Due to changes in course schedule, it is fair game on this year's midterm.
- 50 minutes was given originally to complete this exam, without the use of technology. The only aid provided was the Midterm Exam Reference Sheet, which is on the website, and you will also get during the midterm exam this year!
- Good luck and happy proving!

1. Let p, q, and r be propositions.

   (a) (6 points) Find a truth assignment for these propositions that makes the propositional formula $p \to (q \vee r)$ false.

   p: $\surd$ **true** ◯ false
   q: ◯ true $\surd$ **false**
   r: ◯ true $\surd$ **false**

   (b) (8 points) The following truth table shows the truth values for a propositional formula P containing variables p and q. Give an example of a propositional formula that satisfies these truth conditions. (There are many possible answers! You only need to provide one example.)

p	q	P
T	T	T
T	F	T
F	T	F
F	F	T

   Answer: $p \vee \neg q$

2. Translate the sentences below into formulas of first-order logic using the following symbols.

   Note: the sets listed here can be used as domains of quantification. That is, you could write $\forall x : P, \ldots$ to quantify over all CS22 students. You should not use any other domains of quantification.

   - Sets:
     - P: the set of all CS22 students
   - Predicates:
     - $H(x)$: "x hates exams"
     - $L(x)$: "x loves logic"
     - $T(x, y)$: "x and y studied together"

   (a) (8 points) A student who does not love logic studied with a student who does love logic.

   Answer: $\exists xy : P, L(x) \land \neg L(y) \land T(x, y)$

   (b) (8 points) In every pair of study partners there is at least one who hates exams.

   Answer: $\forall xy : P, T(x, y) \to H(x) \lor H(y)$

3. (a) (6 points) Let A and B be sets of natural numbers. Suppose that 4 is *not* an element of A.

   Which of the following statements could *possibly* be true? Select all that apply.

   - $\surd$ $A \cap B \subseteq A \cup B$
   - $\surd$ $A \cup B \subseteq A$
   - ◯ $4 \in B \setminus \overline{A}$

   (b) (8 points) Give a *counterexample* to the following claim: for any two sets of natural numbers C and D, $C \setminus D \subseteq C \cap D$. Explain why your answer is a counterexample in one or two short sentences.

Solution:
As a counterexample, let $C = \mathbb{N}$, and let $D = \{1\}$.

By definition, that means $C \setminus D$ will contain everything in $\mathbb{N}$ except for 1.

On the other hand, $C \cap D$ would only contain the intersection of the two sets, or 1.

The natural number 2 would be in $C \setminus D$ (as it contains all natural numbers but 1), but it is not in $C \cap D$ (as, that only contains the number 1). Thus, because there is an element in $C \setminus D$ that is not in $C \cap D$, $C \setminus D$ cannot be a subset of $C \cap D$, and our counterexample proves that the claim is false.

4. Let X be the set of integers $\{-1, -2, \ldots, -9, -10\}$ and Y be the set of integers $\{1, 2, \ldots, 9, 10\}$.

(a) (4 points) Define a bijection $f : X \to Y$.

Answer: $f(k) = |k|$ **or** $f(k) = -k$

(b) (6 points) Prove that the function you give in part (a) is a bijection.

Solution:
For $f(k) = |k|$:

f is injective: suppose $a_1, a_2 \in X$ with $f(a_1) = f(a_2)$. Becuase $f(a_1) = |a_1|$ and $f(a_2) = |a_2|$, we substitute, and affirm $|a_1| = |a_2|$. Since both are negative (as they belong in X, which only has negative numbers), we can further determine that, $-a_1 = -a_2$, so $a_1 = a_2$. Because we have shown that for arbitrary $a_1, a_2 \in X$, $f(a_1) = f(a_2) \to a_1 = a_2$, f is injective.

f is surjective: suppose some arbitrary $b \in Y$. Then $-b \in X$ (based on the definitions of the sets), and $f(-b) = |-b| = b$. Thus, every $b \in Y$ (the codomain of Y) is mapped to by something through the function, and f is surjective.

Because f is injective and surjective, f is bijective.

5. I intend to prove the following theorem: every positive integer can be written as the sum of one or more distinct powers of 2. By *distinct*, I mean that each power of two can appear at most once in the sum. (Otherwise, I could write any natural number n as $2^0 + 2^0 + \ldots + 2^0$, where 2^0 is repeated n times.)

   For example, the natural number 5 can be written as $2^2 + 2^0$. 6 can be written as $2^2 + 2^1$.

   I proceed by *strong induction*, with the predicate $P(n)$, "n can be written as the sum of distinct powers of 2."

   (a) (6 points) State and prove the *base case* for this induction argument.

   > **Solution:**
   > In the base case, we show $P(1)$, 1 can be written as the sum of distinct powers of 2. This holds because $1 = 2^0$.

   (b) (6 points) Set up the inductive case for this induction argument. That is: clearly state the induction hypothesis and the proposition that must be proved in order to complete the argument. You do *not* need to provide this proof.

   > **Solution:**
   > Our induction hypothesis is that $P(k)$ holds for all $1 \leq k \leq n$: that is, every positive integer less than or equal to n can be written as the sum of distinct powers of 2.
   > We want to show that $n + 1$ can be written as the sum of distinct powers of 2.

6. Let $R(x, y)$ be a binary relation on the natural numbers, such that $R(x, y)$ is true if $\gcd(x, y) = 1$ and false otherwise.

   (a) (3 points) Is the relation $R(x, y)$ *reflexive*?

   ◯ Yes √ **No**

   (b) (3 points) Is the relation $R(x, y)$ *symmetric*?

   √ **Yes** ◯ No

   (c) (3 points) Is the relation $R(x, y)$ *transitive*?

   ◯ Yes √ **No**

   (d) (5 points) Justify your answer to part (c) in a few short sentences.

   Solution:
   We can prove this with a counterexample.

   $R(2, 3)$ holds, and $R(3, 4)$ holds, but $R(2, 4)$ does not hold, as $gcd(2, 4) = 2$. So, the relation R is not transitive. (Because it fails the transitivity property of $R(a, b) \wedge R(b, c) \to R(a, c)$.)
   More generally, a relation that is symmetric but not reflexive cannot be transitive.