CS22 Discrete Structures and Probability Practice Midterm Exam

Please read these directions carefully before you begin!

- This is based on an actual exam from CS22 last year.
- The practice midterm is not a replacement for reviewing this year's lectures, recitations, and homeworks.
- Modular arithmetic is not on this midterm. Due to changes in course schedule, it is fair game on this year's midterm.
- Write your name and Brown ID number (or some unique identifier) on this cover sheet, and only your Brown ID on subsequent sheets. This will help us keep track of the pages of your exam while grading anonymously.
- 50 Minutes was given originally to complete this exam, without the use of technology. The only aid provided was the Midterm Exam Reference Sheet, which is on the website, and you will also get during the exam.
- Good luck and happy proving!

1. Let p, q, and r be propositions.

   (a) (6 points) Find a truth assignment for these propositions that makes the propositional formula $p \rightarrow (q \vee r)$ false.

p:	◯ true	◯ false
q:	◯ true	◯ false
r:	◯ true	◯ false

   (b) (8 points) The following truth table shows the truth values for a propositional formula P containing variables p and q. Give an example of a propositional formula that satisfies these truth conditions. (There are many possible answers! You only need to provide one example.)

p	q	P
T	T	T
T	F	T
F	T	F
F	F	T

   Answer: ________________________________

2. Translate the sentences below into formulas of first-order logic using the following symbols.

   Note: the sets listed here can be used as domains of quantification. That is, you could write $\forall x : P, \ldots$ to quantify over all CS22 students. You should not use any other domains of quantification.

   - Sets:
     - P: the set of all CS22 students
   - Predicates:
     - $H(x)$: "x hates exams"
     - $L(x)$: "x loves logic"
     - $T(x, y)$: "x and y studied together"

   (a) (8 points) A student who does not love logic studied with a student who does love logic.

   Answer: ______________________________

   (b) (8 points) In every pair of study partners there is at least one who hates exams.

   Answer: ______________________________

3. (a) (6 points) Let A and B be sets of natural numbers. Suppose that 4 is *not* an element of A.

   Which of the following statements could *possibly* be true? Select all that apply.

   ◯ $A \cap B \subseteq A \cup B$
   ◯ $A \cup B \subseteq A$
   ◯ $4 \in B \setminus \overline{A}$

   (b) (8 points) Give a *counterexample* to the following claim: for any two sets of natural numbers C and D, $C \setminus D \subseteq C \cap D$. Explain why your answer is a counterexample in one or two short sentences.

4. Let X be the set of integers $\{-1, -2, \ldots, -9, -10\}$ and Y be the set of integers $\{1, 2, \ldots, 9, 10\}$.

   (a) (4 points) Define a bijection $f : X \to Y$.

   Answer: ____

   (b) (6 points) Prove that the function you give in part (a) is a bijection.

5. I intend to prove the following theorem: every positive integer can be written as the sum of one or more distinct powers of 2. By *distinct*, I mean that each power of two can appear at most once in the sum. (Otherwise, I could write any natural number n as $2^0 + 2^0 + \ldots + 2^0$, where 2^0 is repeated n times.)

   For example, the natural number 5 can be written as $2^2 + 2^0$. 6 can be written as $2^2 + 2^1$.

   I proceed by *strong induction*, with the predicate $P(n)$, "n can be written as the sum of distinct powers of 2."

   (a) (6 points) State and prove the *base case* for this induction argument.

   (b) (6 points) Set up the inductive case for this induction argument. That is: clearly state the induction hypothesis and the proposition that must be proved in order to complete the argument. You do *not* need to provide this proof.

6. Let $R(x, y)$ be a binary relation on the natural numbers, such that $R(x, y)$ is true if $\gcd(x, y) = 1$ and false otherwise.

   (a) (3 points) Is the relation $R(x, y)$ *reflexive*?

   ◯ Yes ◯ No

   (b) (3 points) Is the relation $R(x, y)$ *symmetric*?

   ◯ Yes ◯ No

   (c) (3 points) Is the relation $R(x, y)$ *transitive*?

   ◯ Yes ◯ No

   (d) (5 points) Justify your answer to part (c) in a few short sentences.