Disclaimer

The TAs do not know what is on the midterm. The following is our guide for what we believe will be helpful in preparation. We do not have a solution key for this review. We recommend using the practice exam if you would like to self-assess against reference solutions. Good luck!

1 Proof Techniques

1.1 Direct Proof

We directly use our hypotheses to reason that our conclusion is correct.

> **Practice Problem(s)**
>
> 1. Prove the claim that the product of two odd numbers is odd.

1.2 Proof by Cases

Proof by exhaustion, also known as proof by cases, is a method of mathematical proof in which the statement to be proved is split into a finite number of cases and each case is solved to show that, for every possible "angle" in the domain of a claim, we can exhaustively show that the claim can be proved.

> **Practice Problem(s)**
>
> 1. Prove the claim that there exists irrational $x, y \in R$ such that x^y is rational.
>
> 2. Let's agree that given any two people, they have either met or not. If every pair of people in a group has met, we'll call the group a club. If every pair of people in a group has not met, we'll call it a group of strangers.
>
>   Prove that every collection of 6 people includes a club of 3 people or a group of 3 strangers.[a]
>
> [a] *Mathematics for Computer Science* Eric Lehman. 1.7.

1.3 Counterexample

Counterexamples help us prove that a certain claim is not true. A counterexample is a tangible example, that fits appropriately within the domain of a problem, that disproves the claim being made. Note that not every negative statement can be shown by counterexample (e.g., statements of the form "there does not exist…").

However, you **cannot** prove a claim by showing one example of it. Counterexamples are used to *disprove*. (Alternatively, used to prove an inequality, as of sets.)

For example, the claim "all CS22 students like dinosaurs" can be disproved by finding a student who does not like dinosaurs. Finding this counterexample, however, will not prove that no 22 students like dinosaurs.

Practice Problem(s)

1. Prove or disprove the claim that for all sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

2. Prove or disprove via counter example: $\forall x \in \mathbb{Z}, -1 \leq x \leq 1 \rightarrow x^2 = x$.

3. Consider set A as being the set of positive even integers. $(A_1, A_2) \in R$ if $A_1 = 3 \cdot A_2$. Example, $(18, 6) \in R$. Prove via counterexample that $(A_1, A_2) \in R \wedge (A_2, A_3) \in R \not\rightarrow (A_1, A_3) \in R$.

1.4 Contradiction

To prove the *negation* of a statement $\neg p$, we show that it is impossible for p to hold. This is known as *proof by contradiction*. It proceeds as follows:

1. Assume p is true.

2. Given p is true, use a direct proof to obtain a contradiction.

3. Since p being true leads us to a contradiction, p must be false, i.e., $\neg p$ must be true.

Occasionally, we can also use a similar technique to prove a positive (i.e., not negated) statement. *Before using contradiction, see if a direct approach would suffice.*

Here is how we would prove a (positive) proposition p by contradiction:

1. Assume p is not true.

2. Given p is false, use a direct proof to obtain a contradiction.

3. Since p being false leads us to a contradiction, p must be true.

Practice Problem(s)

1. Prove that there is no least positive real number.

2. Consider a set $A = \{a_1, ..., a_n\}$ with cardinality n. Consider $f : \mathcal{P}(A) \rightarrow \{0, 1\}^n$ where $f(X) = s_1 s_2 ... s_n$ and $s_i = 1$ if $a_i \in X$ and $s_i = 0$ if $a_i \notin X$.

   Prove the claim that if $f(X_1) = f(X_2)$ then $X_1 = X_2$.

Samples of different proof types can be found in the resources section of the 22 website.

2 Logic

2.1 Preliminary Definitions

1. A **propositional formula** is a condensed representation of a truth table using logical operators and variables. We call a propositional formula a *proposition* for short.

2. The term **logical expression** is often used synonymously with the word proposition.

3. Two propositions are **logically equivalent** when they represent the same truth table. We can prove propositions are logically equivalent by either comparing their truth tables or using logical rewrite rules. A full list of the rules you can use is on our course website.

4. A **valid proposition** is one that evaluates to true on any choice of inputs; it is true no matter what. It is also sometimes called a tautology. The classic example of a valid proposition is $b \vee \neg b$ (thanks, Shakespeare).

5. A proposition is **satisfiable** if it evaluates to true on *some* choice of inputs; that is, that there is some assignment of the input variables to true and false that makes the proposition true.

6. A proposition is **unsatisfiable** if it is false on any choice of inputs; it is false no matter what. It is also sometimes called a contradiction. The classic example of an unsatisfiable proposition is $p \wedge \neg p$.

Let's now review the interpretation of each of the following logical operators:

P	Q	P	$P \wedge Q$	$P \vee Q$	$P \oplus Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

2.2 Implication

In the formula $P \rightarrow Q$, we call P the **hypothesis** and Q the **conclusion**. $P \rightarrow Q$ is logically equivalent to $\neg P \vee Q$. In words, this means that for $P \rightarrow Q$ to be true, Q must be true or P must be false.

This choice can seem a little strange at first. Why is $P \rightarrow Q$ true when P is false? Consider the following statement: "If it is raining, I will bring my umbrella." Here are the events that could possibly occur.

- It rains, and I bring my umbrella. That seems fine. The statement is consistent with the situation.

- It rains, and I don't bring my umbrella. The statement does not fit with the situation.

- It doesn't rain, and I bring my umbrella. This situation doesn't seem to directly conflict with the statement. After all, what if I brought my umbrella to block the sun instead? As a result, we say the statement is still consistent with the situation.

- It doesn't rain, and I don't bring my umbrella. The statement seems consistent with this situation, too.

The only scenario where the statement doesn't fit is the second, which is why $P \to Q$ is only false when P is true and Q is false.

- $\neg Q \to \neg P$ is called the **contrapositive** of $P \to \neg Q$ and is logically equivalent. As a result, we have a useful proof technique: to prove the statement "if p, then q" we can instead prove "if not q, then not p."

- $Q \to P$ is called the **converse** of $P \to Q$. It is **not** logically equivalent to $P \to Q$. If both a statement and its converse are true, then the biconditional $P \leftrightarrow Q$ is true.

2.3 Normal Forms

We say a proposition is in **DNF (disjunctive normal form)** when it is the disjunction (clauses ORed together ($\vee$)) of conjunctions (literals ANDed together ($\wedge$)).

We say a proposition is in **CNF (conjunctive normal form)** when it is the conjunction (clauses ANDed together ($\wedge$)) of disjunctions (literals ORed together ($\vee$)).

Here's a truth table, and propositions in DNF and CNF that represent it:

P	Q	R	?
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

DNF: $(P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

CNF: $(\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$

If we have an arbitrary truth table, here are two ways we can think about describing it:

- Listing the true rows.

- Listing the false rows.

Since every row must be either true or false, both of these ways will uniquely describe our truth table.

These two ways correspond to DNF and CNF, respectively. To write a proposition in DNF, we can think about it like this: we find all rows where our proposition should evaluate to true, and we say that we must be in one of those rows. On the other hand, to write a proposition in CNF, we find all rows where our proposition should evaluate to false, and say we are not in any of those rows.

For DNF, we $\wedge$ the true variables and negations of the false variables (to be in the row, the inputs must exactly correspond to the row). For CNF, we $\vee$ the false variables and the negations of the true variables (to not be in the row, we just need at least one variable to be different).

In this way, we can represent any truth table in DNF or CNF. We can also rewrite any logical expression to be in DNF or CNF.

Practice Problem(s)

1. Suppose we define a new operation $\star$ on logical propositions such that

$$x \star y \equiv \neg(x \wedge y)$$

   Create a truth table for each of the following expressions, and state which logical operator the expression is equivalent to.

   - $x \star x$
   - $(x \star y) \star (x \star y)$
   - $(x \star x) \star (y \star y)$
   - $(x \star (x \star y)) \star (y \star (y \star x))$

2. Write two propositions corresponding to the following truth table: one in DNF and one in CNF.

P	Q	R	?
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

2.4 First-Order Logic

In propositional logic, we only consider "atomic" propositions, represented by propositional variables like p and q. First-order logic is more expressive: it allows us to write propositions *about* particular data (like numbers).

In particular, first-order logic lets us write expressions that make assertions about particular entities. Such expressions are called **predicates**; you can think of these a bit like functions from the data in question to the values "true" and "false." For instance, we might define a predicate $\text{Odd}(n)$ that holds of a natural number $n : \mathbb{N}$ if and only if n is odd. Predicates are syntactically represented by *predicate variables* (like Odd), with the value of which they are being asserted written in parentheses after the predicate variable.

A given predicate can only make assertions about a certain *kind* of object: for instance, it wouldn't really make sense to apply the predicate Odd above to an irrational number like $\sqrt{2}$. We therefore define for each predicate a **domain**, the collection of all the possible values of which the predicate can be asserted. (The domain of Odd would be $\mathbb{N}$.)

Given some predicate, we may wish to make claims about whether it holds of *any*, or of *all*, elements in its domain. **Quantifiers** allow us to express such claims in first-order logic. There are two quantifiers of note:

1. Universal quantifier: denoted by the $\forall$ symbol, it represents that a predicate holds for *every* element in its domain.

2. Existential quantifier: denoted by the $\exists$ symbol, it represents that a predicate holds for *some* element in its domain.

For instance, the formula $\forall n : \mathbb{N}, \text{Odd}(n)$ asserts that every natural number is odd (this is false!). On the other hand, $\exists n : \mathbb{N}, \text{Odd}(n)$ asserts that at least one odd natural number exists (this is true!).

Note that a universal quantification over an empty domain is always true, while an existential quantification over an empty domain is always false.

We can also chain quantifiers in sequence to represent a more complex proposition.

Example

Problem: Render *Goldbach's conjecture*, that every integer greater than 2 is the sum of two primes, in first-order logic.[a]

We first can reformulate this in English in a way that better matches our first-order syntax: "For every even integer n greater than 2, there exist primes p and q such that $n = p + q$". *Note: the TAs find this to be especially helpful!*

We can then define some predicates. Let G be the predicate on natural numbers defined by $G(n) := n \geq 2$; that is, $G(n)$ is true just in case $n \geq 2$. Let P be the predicate on natural numbers such that $P(n)$ holds just in case n is prime.

We can thus describe the conjecture in first-order logic as follows:

$$\forall n : \mathbb{N}, G(n) \to \exists p, q : \mathbb{N}, P(p) \wedge P(q) \wedge n = p + q$$

Note that the order of quantifiers is essential. If we switched the order of the quantifiers, we would essentially assert that there are two prime numbers whose sum is equal to every number greater than 2. (This is clearly false!)

Note also the different ways we "restrict" universal and existential quantifiers (so that we are only considering n satisfying G and so that the witnesses p and q must have property P). If we switched the $\to$ and $\wedge$ symbols in the above, our formula would be incorrect! (Think about why.)

[a] *Mathematics for Computer Science* Eric Lehman. 3.6.

Practice Problem(s)

1. For following questions, assume these definitions:

   - Sets:
     - T: The set of CS22 TAs.
   - Predicates:
     - $D(x)$ "x double majors at Brown"
     - $M(x, y)$ "x and y share a major in common."
     - $F(x, y)$ "x and y are friends."
   - Functions
     - $n(x)$: The number of majors x studies.
   - Constants
     - r: Rob, cs22 professor (also known as the Last Logician)!

   Using the above definitions, translate these sentences into First-Order logic.

   a) Every TA on the 22 staff double majors.

   b) A TA on staff is friends with every other TA on staff.

   c) Every TA on staff is friends with Rob. (<3).

   d) TAs that study the same major are friends.

e) Every TA has a friend who studies more majors than them.

f) There is a TA who doesn't have the same major as anyone else on staff.

g) There is a TA that studies more majors than any other TA on staff.

2. Translate the following sentences into first-order logic. You may only use $\mathbb{N}$ as a domain of quantification. You may use the relations $=$, $<$, $\leq$, $>$, and $\geq$; functions $+$, $-$, and $\times$; and one-place predicate Prime. You may not use any quantifiers or connectives other than those we have discussed in this guide.

   a) The difference of any two natural numbers is no greater than their sum.

   b) No prime number is square.

   c) (Challenge) There is a *unique*[a] natural number that is less than every other natural number.

[a] I.e., it is the only natural number with this property.

3 Formal Proofs

Remember that the Lean documentation website has additional materials covering these topics!

3.1 Proof Structure

When we write a proof, we are implicitly (or, in the case of Lean, explicitly) manipulating a **proof state**. Our proof state consists of:

1. One or more **goals**, the propositions we are trying to prove; and
2. Attached to each goal, a **context** consisting of **hypotheses**. These are things that we know or have assumed *for that particular goal*.

Our proof begins with no hypotheses and one goal (the theorem statement we're trying to prove). A proof proceeds by applying **proof rules** to our proof state. A proof rule may modify our goal, close (i.e., fully prove) our goal, add new goals, modify a hypothesis, or add new hypotheses.

When a new goal is created (e.g., as a result of a proof rule like disjunction elimination), it gets it own, distinct context. Changes to the context of one goal do *not* appear in the context of another! Proof rules apply only to one goal at a time! We "focus" one goal to work on, moving onto the next after closing the previous one.

3.2 Propositional Proof Rules

We divide our proof rules into two main categories: introduction rules and elimination rules.

1. **Introduction rules** tell you what is required in order to *prove* a *goal* of a given form.

2. **Elimination rules** tell you what is required in order to *extract information from* a *hypothesis* of a certain form.

In other words, if your goal contains some connective, you can use an introduction rule to "simplify" your goal into the propositions required in order to obtain the larger formula with the connective. Note that the vocabulary is a little confusing here: once you apply an introduction rule, the relevant connective "disappears" from your goal. This is because applying an introduction rule works *backward*: we're saying that in order to conclude a goal with the relevant connective, it suffices to show some simpler goal(s) that don't include that connective.

On the other hand, elimination rules "extract data from" *hypotheses* that you already know. They tell you what "simpler" propositions you're allowed to conclude based on the knowledge that you know a proposition containing a certain connective. Here are the proof rules for each propositional connective:

Connective	Intro Rule(s)	Elim Rule(s)
Conjunction ($\land$)	To show $p \land q$, show p, then show q (Lean: `split_goal`).	If you know $p \land q$, conclude p and q (Lean: `eliminate`).
Disjunction ($\lor$)	- To show $p \lor q$, show p (Lean: `left`). - To show $p \lor q$, show q (Lean: `right`).	(Proof by cases) If you know $p \lor q$, and you can show r assuming p, and you can show r assuming q, then conclude r (Lean: `eliminate`).
Implication ($\to$)	To show $p \to q$, assume p and show q (Lean: `assume`).	(*Modus ponens*) If you know $p \to q$ and you know p, conclude q (Lean: `have`/`apply`).
Bi-implication ($\leftrightarrow$).	To show $p \leftrightarrow q$, show $p \to q$, then show $q \to p$ (Lean: `split_goal`).	If you know $p \leftrightarrow q$, conclude $p \to q$ and $q \to p$ (Lean: `eliminate`).
Negation ($\neg$)	(Proof by contradiction) To show $\neg p$, assume p and derive a contradiction (Lean: `assume`).	(Explosion) If you know p and you know $\neg p$, conclude anything (Lean: `contradiction`).

We also have the "structural" rule of **atom introduction**: if you know p, then you can conclude p (Lean: `assumption`).

3.3 First-Order Proof Rules

In first-order logic, we enrich our notion of a context to allow for *data*. In addition to hypotheses, our context tracks variables that correspond to things like numbers or sets.

Just as for connectives, we have introduction and elimination rules for quantifiers:

Quantifier	Intro Rule	Elim Rule
Universal ($\forall$)	To show $\forall x : T,\ P(x)$, fix an arbitrary $t : T$ in your context, then show $P(t)$. (Lean : `fix`)	If you know $\forall x : T,\ P(x)$ and you have a piece of data $t : T$, then conclude $P(t)$. (Lean: `have`)
Existential ($\exists$)	To show $\exists x : T,\ P(x)$, choose a *specific* witness value $k : T$, then show that $P(k)$ holds. (Lean: `existsi`)	If you know $\exists x : T,\ P(x)$, add a piece of data $t : T$ to your context and conclude $P(t)$. (Lean: `eliminate`)

We have also seen two other kinds of rules for first-order logic:

1. **Rewriting** allows us to substitute equals for equals in a goal or hypothesis. For instance, if we have a hypothesis that says that $x = y$ and our goal is $P(x)$, rewriting lets us turn our goal into $P(y)$. (Lean: `rewrite`)

2. **Arithmetical reasoning** lets us close goals that are solvable using straightforward arithmetic or high-school algebra, like $15 + 151 > 22$ or (assuming $x, y : \mathbb{N}$) $x + y > x$. (Lean: `numbers`, `linarith`, `polyrith`)

Practice Problem(s)

1. Prove the proposition $(p \vee q) \to (\neg p \to q)$ using proof rules. Carefully describe the proof rule you are applying at each step and how it updates your proof state. If multiple goals arise, clearly indicate which goal you are focusing on and when you move from one goal to the next.

2. Suppose that we are in the middle of a proof, and our proof state looks as follows (assume that this is the only goal):

$$
\begin{aligned}
&p\ q\ r && : \mathsf{Prop} \\
&hpq && : p \wedge q \\
&hp && : p \\
&hq && : q \\
&hpr && : p \to r \\
&hr && : r \\
&\vdash q \wedge r
\end{aligned}
$$

What theorem might we be trying to prove? (In other words, what could our original goal at the start of the proof have been?) Based on this, what proof rules must we have applied to get to our current state? What proof rules are required to finish this proof, and how will each change our proof state?

3. Suppose we've defined certain natural numbers to be *prehistoric*. (We won't actually specify what we've defined this term to mean.) Translate the sentence "for every prehistoric natural number, there's some some non-prehistoric natural number that is greater than it." The only domain of quantification you should use is $\mathbb{N}$; the only predicate symbols you should use are $\mathsf{Prehistoric}$ and $>$.

   Explain what proof rules you would use to prove this proposition. What would your proof state look like just after applying the introduction rule for the second quantifier in your formula?

4. Suppose we decided to change our formal proof system by getting rid of the usual elimination rule for disjunction and replacing it with the following: "If you know $p \vee q$, conclude p." Give an intuitive explanation of why this would be a bad idea. Then show (by exhibiting an appropriate proof) that we can use this formal proof system to prove False.

5. Why do we require that every goal have a distinct context? Give an example of a connective that highlights the importance of this property.

   Suppose, contrarily, that our proof system were such that all goals shared the same context (i.e., if a hypothesis is added to one goal's context, it gets added to all other goals' contexts, too). Give a proof that $0 = 1$ using this system.

4 Sets and Notation

A set is a collection of objects without order or repetition.

4.1 Membership vs. Subsets

If an object s is a member of a set S, we say $s \in S$. If a set T is a subset of a set S, we write $T \subseteq S$. This means that every member of T is also a member of S.

Practice Problem(s)

1. A is any set. Which of the following is **always true**?

   i. $A \subseteq A$

   ii. $\{\} \subseteq A$

   iii. $\{\} \in A$

2. A is any set and $\mathcal{P}(A)$ is the set of all subsets of A. Which of the following is **always true**?

   i. $A \in \mathcal{P}(A)$

   ii. $A \subseteq \mathcal{P}(A)$

   iii. $\emptyset \in \mathcal{P}(A)$

   iv. $\emptyset \subseteq \mathcal{P}(A)$

   v. $\{A, \emptyset\} \subseteq \mathcal{P}(A)$

3. S is the set of students in CS22. B is the set of students at Brown. Duncan is a student in CS22. Which of the following is **always true**?

   i. $S \subseteq B$

   ii. Duncan $\subseteq S$

   iii. Duncan $\in S$

   iv. $\{\text{Duncan}\} \subseteq B$

4.2 Set Operations

- The union $A \cup B$ of two sets A and B is the set of all elements that are in A or B.
- The intersection $A \cap B$ of two sets A and B is the set of all elements that are in A *and* B.
- The set difference $B \setminus A$ of two sets A and B is the set of all elements that are in B, but that are not in A.
- The complement $\overline{A}$ of a set A is the set of all elements that are *not* in A (where "all elements" refers to all elements in some universal set U.)
- The cardinality $|A|$ of a set A is the number of elements of A. Remember that sets have no duplicates!

Practice Problem(s)

1. For this problem, let

$$\begin{aligned}
A &= \{-3, -1, 0, 6\} \\
B &= \{x : \mathbb{Z} \mid x^2 \leq 5\} \\
C &= \{x : \mathbb{Z} \mid \exists\, y \in \mathbb{Z} \text{ s.t. } 3y = x\} \\
D &= \{x : \mathbb{Z} \mid \exists\, y \in \mathbb{Z} \text{ s.t. } y^2 <= x\}
\end{aligned}$$

Find the following sets (not all sets are finite):

i. $A \cup B$

ii. $A \cap \overline{C}$

iii. $B \cap D$

iv. $A \setminus C$

v. $B \setminus (C \cup D)$

vi. $C \cap D$

vii. $C \cup D$

Find the cardinalities of the followings sets:

i. A

ii. $(B \setminus C) \setminus D$

iii. $(A \cup B) \cap (C \cap D)$

4.3 Power Sets

The *power set* of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S. The power set of S has cardinality $2^{|S|}$. We proved this last result by noticing that there are the same number of subsets of a set of size n as there are binary strings of length n (see the sample bijective proof on the website).

Practice Problem(s)

1. Let A, B, C, D be the sets from above, find the following sets:

   i. $\mathcal{P}(A \cap B)$

   ii. $\mathcal{P}(A) \cap \mathcal{P}(D)$

   iii. $\mathcal{P}(B) \setminus \mathcal{P}(C)$

   iv. $\mathcal{P}(\mathcal{P}(\varnothing))$

2. Prove that $|A| < |\mathcal{P}(A)|$ for any arbitrary finite set A.

4.4 Product

The product of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) for $a \in A$, $b \in B$. The product of a single set, A, is the set of all ordered pairs (a, a) where $a \in A$.

Practice Problem(s)

1. Let A, B, C, D be the sets from above, find the following sets:

   i. $A \times B$

   ii. $B \times \{\emptyset\}$

   iii. $C \times \emptyset$

   iv. $\emptyset \times \emptyset$

2. Prove that $(\mathbb{Z} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{Z}) = \mathbb{N} \times \mathbb{N}$.

3. Disprove the following claim: $|A \times A| < |\mathcal{P}(A)|$ for any arbitrary finite set A.

4. Disprove the following claim: for any two finite sets A and B, $|\mathcal{P}(A \times B)| = |\mathcal{P}(A) \times \mathcal{P}(B)|$.

4.5 Proof by Set-Element Method

How do you prove that some set A equals some set B? First show that $A \subseteq B$ and then you show that $B \subseteq A$. If every element in A is also an element in B and every element in B is also an element of A, then A must equal B.

To show that $A \subseteq B$ you consider an arbitrary element in A and show it is also in B. In use, this looks like the following:

1. Let x be an element of set A.

2. Prove that x is also an element of B.

3. Conclude that $A \subseteq B$.

Example

Claim: $A \cap (A \cup B) = A$.

Proof. We show that both $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$.

We first prove the subclaim that $A \cap (A \cup B) \subseteq A$.

Consider any $x \in A \cap (A \cup B)$. By definition of intersection, this means that $x \in A$ and $x \in A \cup B$. Because every arbitrary x in $x \in A \cap (A \cup B)$ is in A, we can conclude that $A \cap (A \cup B) \subseteq A$.

We then prove the subclaim that $A \subseteq A \cap (A \cup B)$.

Consider any $x \in A$. By definition of union, we can reason that $x \in A \cup B$. Because we know that $x \in A \land x \in (A \cup B)$, by definition of intersection, we conclude $x \in A \cap (A \cup B)$. It follows that because every arbitrary x in A is in $A \cap (A \cup B)$, $A \subseteq A \cap (A \cup B)$.

Therefore, by the set element method we have proved that $A \cap (A \cup B) = A$. □

Practice Problem(s)

1. Prove the claim that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ via the set element method.

2. Prove the following properties using the set-element method.

   a) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

   b) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

3. Prove or disprove each of the following. To prove equality, use the set-element method.

   a) $A \cup (B \cap C) = (A \cup B) \cap C$

   b) $\overline{A} \cup (A \cap B) = \overline{A} \cup B$

   c) $A \cap (B \setminus C) = (A \setminus B) \cap (A \setminus C)$

4.6 Set Algebra

1. Conversion of one side of the equation to the other (or conversion of both sides to an identical expression) using stated laws of set algebra. (See list of set identities on course website!)

2. Conclusion based on the biconditionality of the steps taken.

Note: Do not assume equality before applying set identities! Either rewrite one side to look like the other or rewrite both sides separately to look like the same expression.

Example

Claim: $(A \cap B) \cup (A \setminus B) = A \cap (B \cup (A \setminus B))$.

Proof.

$$
\begin{aligned}
&(A \cap B) \cup (A \setminus B) && \\
&= (A \cap B) \cup (A \cap \overline{B}) && \text{(Set Difference Law)} \\
&= A \cap (B \cup \overline{B}) && \text{(Distributive Law)} \\
&= A \cap U && \text{(Complement Law)} \\
&= A && \text{(Identity Law)} \\
&= A \cap (A \cup B) && \text{(Absorption)} \\
&= A \cap (B \cup A) && \text{(Commutativity)} \\
&= A \cap ((B \cup A) \cap U) && \text{(Identity Law)} \\
&= A \cap ((B \cup A) \cap (B \cup \overline{B})) && \text{(Complement Law)} \\
&= A \cap (B \cup (A \cap \overline{B})) && \text{(Distributive Law)} \\
&= A \cap (B \cup (A \setminus B)) && \text{(Set Difference Law)}
\end{aligned}
$$

□

Practice Problem(s)

1. $(A \cup B) \cap \overline{(A \cap B)} = (B \setminus A) \cup (A \setminus B)$.
2. $(A \cap \overline{B}) \cup B = A \cup B$.
3. $(A \setminus B) \setminus (B \setminus C) = (A \cup B) \setminus (A \cap B)$.

5 Relations

5.1 Cartesian Products

A *binary* (or *two-place*) *relation* R consists of a set A, called the *domain*; a set B, called the *codomain*; and a subset of the Cartesian product $A \times B$ called the *graph*. If we say that a relation R is *on* a set A, we mean that both its domain and codomain are A.

Always remember to specify the set(s) on which the relation is defined!

We write aRb or $(a, b) \in R$ to mean that a is related to b by R.

5.2 Reflexivity

A relation R on A is *reflexive* if for all $a \in A, (a, a) \in R$. In other words, a relation is reflexive if *every element* in the set A is related to itself in R. This is why it's important to specify a set when talking about a relation: you can't tell if a relation is reflexive if you don't know which elements have to be related to themselves (and every element must be!).

5.3 Symmetry and Transitivity

A relation R is *symmetric* if for all a, b in its domain, the following holds: **if** $(a, b) \in R$, **then** $(b, a) \in R$.

A relation R is *transitive* if for all a, b, c in its domain, the following holds: **if** $(a, b) \in R$ and $(b, c) \in R$, **then** $(a, c) \in R$. Remember that a, b, and c do not need to be different elements.

It's important to note that the definitions of symmetry and transitivity are phrased as if-then statements. A relation is symmetric/transitive *unless* it violates the appropriate if-then condition. To violate the condition, you must simultaneously satisfy the if-clause, and violate the then-clause.

Consider the following example of a relation that is not transitive: the ordered pairs $(1, 2)$ and $(2, 1)$ are in the relation (this satisfies the if-clause of the transitivity definition) but there is no pair $(1, 1)$ in the relation (this violates the then-clause).

As another illustrative example: any empty relation is reflexive, symmetric, and transitive, as there are no ordered pairs in the empty relation to satisfy the if-clause of the definitions.

5.4 Equivalence Relation

An *equivalence relation* is a relation that is reflexive, symmetric, and transitive.

5.5 Equivalence Classes

Let R be an equivalence relation on A. Then the *equivalence class* of $a \in A$ is defined as

$$[a]_R := \{x \mid x \in A, (x, a) \in R\}.$$

Note that a is not unique (unless it is the only element in its equivalence class.) Rather, any element in the same equivalent class can serve equally well as the representative for the class. An equivalence relation splits a set into equivalences classes. In other words, it forms partitions.

A *partition* of a set A is a collection of nonempty subsets $B_1, \ldots, B_k$ of A such that

1. $B_1 \cup \cdots \cup B_k = A$, and
2. $B_i \cap B_j = \emptyset \;\; \forall i, j$ where $i \neq j$.

Practice Problem(s)

1. Consider the set B of all students at Brown. For each of the following relations on B, state if they are reflexive, symmetric, or transitive. If they are an equivalence relation, then list the equivalence classes.

   i. Two students are related if they are the same age (e.g. 21).

   ii. s_1 and s_2 are students and $(s_1, s_2) \in R$ if s_1 is younger than s_2.

   iii. Two students are related if they are studying anthropology.

   iv. Two students are related if they go to Brown.

2. Let $A = \{1, 2, 3\}$. Consider the following relations on $\mathcal{P}(A)$. State if they are reflexive, symmetric, or transitive. If they are an equivalence relation, then list the equivalence classes.

   i. $(S_1, S_2) \in R$ if $|S_1| = |S_2|$.

   ii. $(S_1, S_2) \in R$ if $S_1 \subseteq S_2$.

   iii. $(S_1, S_2) \in R$ if S_1 and S_2 share an element.

6 Functions

6.1 Formal Definition

A *function* $f : A \to B$ is a relation on A and B with the following property: for every $a \in A$ there exists exactly one pair (a, b) in the relation, where $b \in B$.

We call A the domain and B the codomain.

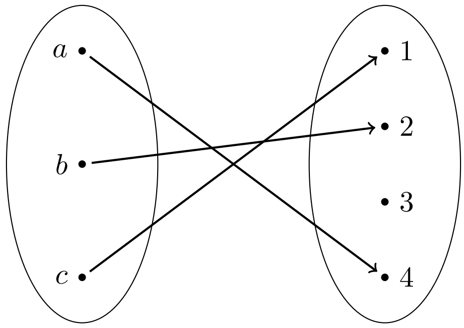
It's important to note that a function is characterized not only by the "rule" that maps inputs to outputs, but also by the domain and codomain.

Additionally, we call the set of all $b \in B$ such that there exists $a \in A$ where $f(a) = b$ the *image* of f. In other words, the image is the set of all elements mapped to by f.

6.2 Injectivity

A function is injective if for all $b \in B$, there exists at most one $a \in A$ such that $f(a) = b$. In other words, no two distinct elements map to the same thing!

If a function $f : A \to B$ is injective, we know that $|A| \leq |B|$. This is because every element in A needs some unmatched element in B, so B needs to have at least as many elements as A!



There are two ways to prove that a function is injective:

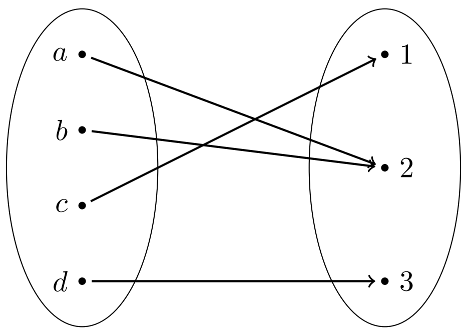
1. Consider two arbitrary elements a and b of the domain, and show that if $f(a) = f(b)$, then we must have $a = b$.

2. Consider two arbitrary distinct elements $a \neq b$ in the domain. Show that they must map to distinct outputs $f(a) \neq f(b)$.

6.3 Surjectivity

A function is surjective if for all $b \in B$, there exists at *least* one $a \in A$ such that $f(a) = b$. In other words, no element in the codomain gets left behind: there is always some element that maps to it. Equivalently, a function is surjective if the image of the function is the entire codomain.

If a function $f : A \to B$ is surjective, we know that $|A| \geq |B|$. This is because every element in B needs some element in A to map to it, so A needs to have at least as many elements as B.



To prove that a function is surjective, consider an arbitrary element in the codomain, and construct the specific element in the domain that maps to it.

Practice Problem(s)

1. Let $S = \{0, 1\}$, $T = \{t \mid t \subseteq S \times S\}$, and R be the set of all possible functions from S to S. [a]

   i. Can an injection from T to R exist? If so, give one such injection and prove that this mapping is indeed injective. If not, prove why such a mapping cannot exist.

   ii. Can a surjection from T to R exist? If so, give one such surjection and prove that this mapping is indeed surjective. If not, prove why such a mapping cannot exist.

[a] *HW3 Problem 1* CSCI0220 2022 Spring

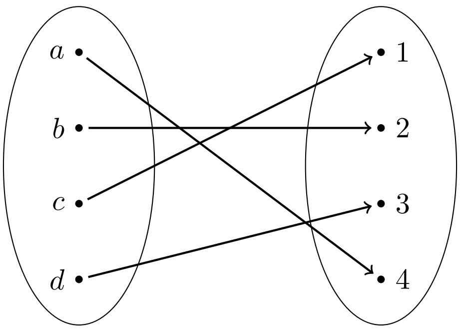
6.4 Bijectivity

A bijection is a function that is both injective and surjective. Thus, to prove that a function is a bijection, prove that it is injective and surjective.

If we combine our results from injectivity and surjectivity, we know that the cardinality of the domain must be less than or equal to that of the codomain (by injectivity), and that the cardinality of the domain must be greater than or equal to that of the domain (by surjectivity.) Thus, the cardinalities of the two sets must be equal. This is a powerful result:

There exists a bijection between two sets if and only if they have equal cardinality.

Thus, to prove that the sizes of two sets are equal, it suffices to prove that there exists a biejction between them.



Practice Problem(s)

1. For each of the following, state if it is a function- if it is a function, conclude if it is injective and/or surjective.

   a) $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$

   b) $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ where $f(x) = x^2$. $\mathbb{Z}^+$ denotes the positive integers.

c) $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = \sqrt{x}$.

d) $f : A \to B$, where $f(\text{student})$ = the dorm that the student lives in, A represents the set of first year students at Brown, and B represents the set of first year dorms at brown.

e) $f : A \to B$, where $f(\text{student})$ = the banner ID of student, A represents the set of students at Brown, and B represents the set containing the Banner IDS of all current students at brown.

f) f : People in the World $\to \{0, 1\}$ where $f(\text{person}) = 1$ if they are Prof. Lewis and 0 otherwise.

g) $f : A \to \mathbb{Z}$, where A represents the set of libraries at Brown and $f(\text{Library})$ = number of books in the library.

h) $f : S \to \mathcal{P}(S)$ where $f(S) = \{S\}$.

i) $f : \mathcal{P}(\{1, 2, 4\}) \to \{0, 1, 2, 3\}$ where $f(X) = |X|$.

2.

3. Let A be a set with n elements. Let T be the set of all ordered pairs (X, Y) where X and Y are subsets of A. Let S be the set of 0/1/2/3 strings of length n. That is, elements of S are strings of length n where each character is 0, 1, 2, or 3. Prove that T and S must be the same size by defining a bijection between T and S. [a]

[a] *HW3 Problem 2* CSCI0220 2023 Spring

7 Induction

7.1 Template and Weak Induction

Induction is a proof method for which we can assume some n case, and prove that every $n + 1$ case holds. If we can prove that the $n + 1$ case holds, we can confirm that our original claim holds for all values of n in the desired domain..

Idea: If you are stuck on an induction problem on the exam, start by writing out the inductive hypothesis and the structure of the proof. You will receive partial credit for this and it will also help you think of how to proceed.

Idea: Often the inductive step is a direct proof using the inductive hypothesis. This is not always the case; sometimes you might have to use ***proof by cases*** *or even* ***contradiction****.*

We will first provide a review of the template for an inductive proof and provide an example.

Example

For example, say we are trying to prove that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ is true for all $n \in \mathbb{N}$.

1. Define the predicate $P(n)$.

   Let $P(n)$ be the predicate that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

2. Show that the base case is true.

   We will first show $P(0)$ is true. $\sum_{i=0}^{0} i = 0$ and $\frac{0(0+1)}{2} = 0$ so they are equal as needed.

3. Assume the inductive hypothesis is true. If you are using stardard induction then you will assume $P(k)$ is true for some integer k. If you are using strong induction then you will assume $P(i)$ is true for all $i \leq k$. Either way, you should specify that k is some integer greater than or equal to your greatest base case.

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$.

4. Show that $P(k+1)$ is true given the inductive hypothesis.

   We will now show that $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

   We know that $\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$.

   By our inductive hypothesis $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

   Therefore

$$
\begin{aligned}
\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

   as needed. $\square$

5. Conclude the proof.

   Therefore, as $P(0)$ is true and $P(k)$ implies $P(k+1)$ for all $k \in \mathbb{Z}$, $k \geq 0$, $P(n)$ is true for all nonnegative integers n.

Practice Problem(s)

1. Show that $6 \mid (n^3 - n)$ for all $n \in \mathbb{N}$.

2. Recall that the Fibonacci numbers are defined by $f_0 = 0, f_1 = f_2 = 1$ and the recursive relation $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. Challenging exercise:

   Show that f_n and f_{n+1} are 'relatively prime' for all $n \geq 1$. That is, they share no factor in common other than the number 1.

7.2 Strong Induction

The difference in approach between weak and strong induction comes in the induction hypothesis! In weak induction, we only assume that the predicate holds for some arbitrary step k, while in strong induction, we assume that the predicate holds at all steps from the base case to some arbitrary step k. Your inductive step may differ depending on whether you approach a problem using weak or strong induction, but they are equivalent!

Practice Problem(s)

1. Define the sequence S as follows: $S_1 = 1$, $S_2 = 3$, $S_n = S_{n-1} * S_{n-2}$ for integers $n \geq 2$. Prove that S_n is odd for all positive integers n.

8 Number Theory

8.1 Definitions

Definition 1: We say that a *divides* b, denoted $a \mid b$, when $b = ka$ for some $k \in \mathbb{Z}$.

Definition 2: We say that a is *congruent* to b mod m, denoted $a \equiv b \pmod{m}$, if $m \mid (b - a)$. Another way to say this is that $a = b + km$ for some $k \in \mathbb{Z}$. Yet another way to say this: a and b have the same remainder upon division by m. Take a moment to convince yourself that these statements are equivalent.

8.2 Properties of Congruence Relations:

For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{m}$,

1. $a + c \equiv b + c \pmod{m}$ for any $c \in \mathbb{Z}$
2. $ac \equiv bc \pmod{m}$ for any $c \in \mathbb{Z}$
3. $a^n \equiv b^n \pmod{m}$ for $n \in \mathbb{Z}^+$

If we also have $c \equiv d \pmod{m}$,

1. $a + c \equiv b + d \pmod{m}$
2. $ac \equiv bd \pmod{m}$

8.3 GCD

The greatest common denominator of a and b is the largest positive integer which divides both a and b. To find the gcd of two numbers, we can run the Euclidean algorithm.

Theorom 1: For any $a, b \in \mathbb{Z}$ there exists $u, v \in \mathbb{Z}$ such that $au + bv = \gcd(a, b)$. In words, we say that a and b can be written as a linear combination of their gcd.

Theorem 2: An integer is a linear combination of a and b if and only if it is a multiple of their gcd.

8.4 Multiplicative Inverse

Consider the particular congruence

$$ax \equiv 1 \pmod{m}.$$

If this equation has a solution, then we know we can find some integer x which, when multiplied by a, yields $1 \pmod{m}$. We define this integer to be the *multiplicative inverse* of $a \pmod{m}$, and we denote it a^{-1}. If a multiplicative inverse exists $\pmod{m}$, then when working $\pmod{m}$, we can "divide" by a—that is, we can multiply two sides of a congruence by a^{-1}, cancelling a from both sides.

When does a multiplicative inverse exist? According to the above Theorem 2: a^{-1} exists if and only if $\gcd(a, m)$ divides 1 (which is c in this particular congruence.) Thus, a^{-1} exists $\pmod{m}$ if and only if $\gcd(a, m) = 1$, that is, if and only if a and m are relatively prime. How do we find the multiplicative inverse? We can run the Euclidean algorithm and then backtrack to obtain the multiplicative inverse (gcdcombo).

8.5 Fermat's little Theorem

If p is prime and does not divide $a \in \mathbb{Z}$ then

$$a^{p-1} \equiv 1 \pmod{p}.$$

This means a^{p-2} is a multiplicative inverse for a mod p.

8.6 Euler's Totient Function

The totient function of n is a count of how many positive integers less than or equal to n are relatively prime to it. For any prime p, $\phi(p) = p - 1$. If m and a are relatively prime, then

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

This means $a^{\phi(m)-1}$ is a multiplicative inverse for a mod m. Fermat's little theorem is just a special case of this rule.

Practice Problem(s)

1. Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a + c \equiv b + d \pmod{m}$
2. Compute the multiplicative inverse of 8 mod 27 with the Euclidean algorithm and the Euler-Fermat method.
3. For all positive (non-zero) integers a, b, and k, prove that $\gcd(ka, kb) = k \cdot \gcd(a, b)$.
4. Use Fermat's Little Theorem or Euler's Theorem to find the inverse of these numbers. If the inverse does not exist, show why it does not exist.

   i. $14^2 \pmod{5}$

   ii. $1452 \pmod{9}$

   iii. $4^6 \pmod{15}$