CS22 Discrete Structures and Probability 2025 Midterm Practice Exam

Please read these directions carefully before you begin!

- This is based on an actual exam from CS22 last year.
- The practice midterm is not a replacement for reviewing this year's lectures, recitations, and homeworks.
- 50 Minutes was given originally to complete this exam, without the use of technology. The only aid provided was the Midterm Exam Reference Sheet, which is on the website, and you will also get during the exam.
- **Disclaimer**: you should not draw any conclusions about this year's exam's length/difficulty/structure from the practice exam.
- Good luck and happy proving!

1. (a) (6 points) Let a, b, and c be propositional variables. Suppose that the formula

$$(\neg(a \to b)) \wedge (\neg b \vee c)$$

is true. What can we conclude about the truth values of a, b, and c?

a:	$\surd$ **must be true**	◯ must be false	◯ could be either
b:	◯ must be true	$\surd$ **must be false**	◯ could be either
c:	◯ must be true	◯ must be false	$\surd$ **could be either**

(b) (6 points) Give an example of a propositional formula containing variables p, q, and r that is *unsatisfiable.*

Answer: $p \wedge \neg p \wedge q \wedge r$

2. Translate the sentences below into formulas of first-order logic using the following symbols.

   Note: the sets listed here can be used as domains of quantification. That is, you could write $\forall x : P, \ldots$ to quantify over all Brown professors. You should not use any other domains of quantification.

   - Sets:
     - P: the set of all Brown professors
   - Predicates:
     - $H(x)$: "x writes hard exams"
     - $J(x)$: "x can tell good jokes"
     - $K(x, y)$: "x knows y"

   (a) (6 points) Every Brown professor knows another Brown professor who cannot tell good jokes.

   Answer: $\forall x : P, \exists y : P, \neg x = y \wedge K(x, y) \wedge \neg J(y)$

   (b) (6 points) Out of all Brown professors, only those who write hard exams can tell good jokes.

   Answer: $\forall x : P, J(x) \rightarrow H(x)$

3. Let X be the set of integers $\{0, 1, 2, \ldots, 11\}$. Let $f : \mathbb{N} \to X$ be the function such that $f(n) = \text{rem}(2n, 12)$, that is, the remainder when $2n$ is divided by 12.

   (a) (6 points) Is f surjective? Justify your answer, stating clearly what it would mean for f to be surjective.

   Solution:
   Surjectivity means that every element in the codomain is mapped to by f from some element in the domain. In logic, that is, $\forall y \in X, \exists x \in \mathbb{N}, f(x) = y$.
   No, f is not surjective. $f(n)$ is always even so it never hits 3.

(b) (6 points) Is f injective? Justify your answer, stating clearly what it would mean for f to be injective.

Solution:
Injectivity means that no two elements of the domain map to the same element of the codomain. In logic: $\forall x, y \in \mathbb{N}, x \neq y \to f(x) \neq f(y)$ or $\forall x, y \in \mathbb{N}, f(x) = f(y) \to x = y$.
No, f is not injective. $f(0) = f(6)$.

4. (a) (4 points) Which one of the following predicates on $\mathbb{N} \times \mathbb{N}$ (with variables x and y) correctly translates the statement "x divides y"?

   - √ $\exists k : \mathbb{N}, y = kx$
   - ◯ $\exists k : \mathbb{N}, x = ky$
   - ◯ $\forall k : \mathbb{N}, y = kx$
   - ◯ $\forall k : \mathbb{N}, x = ky$

   (b) (4 points) Which one of the following predicates on $\mathbb{N}$ (with variable p) correctly translates the statement "p is prime"?

   - ◯ $p > 1 \wedge \forall n : \mathbb{N}, \exists k : \mathbb{N}, kp \mid n$
   - ◯ $\exists q : \mathbb{N}, q > 1 \wedge q \mid p$
   - √ $p > 1 \wedge \forall n : \mathbb{N}, n > 1 \wedge n \mid p \rightarrow n = p$
   - ◯ $p > 1 \wedge \forall n : \mathbb{N}, p \mid n \rightarrow \exists k : \mathbb{N}, n = kp$

5. Consider the following statement: for every positive integer $n \geq 5$, $2^n \geq n^2 + 4$. We would like to prove this by induction from a starting point.

   (a) (2 points) State the induction predicate that you will use in your proof.

   Solution: $P(n) := 2^n \geq n^2 + 4$. Note that n is not universally quantified: this is a predicate with variable n!

   (b) (4 points) State and prove the *base case* for this induction argument.

   Solution:
   In the base case, we show $P(5)$, $2^5 \geq 5^2 + 4$. This holds because $32 \geq 29$.

(c) (6 points) Set up the inductive case for this induction argument. That is: clearly state the induction hypothesis and the proposition that must be proved in order to complete the argument. You do *not* need to provide this proof.

Solution:
Our induction hypothesis is that $P(k)$ holds for some fixed k: that is, $2^k \geq k^2 + 4$. We want to show that it holds for $k+1$, that is, that $2^{k+1} \geq (k+1)^2 + 4$.

6. Let $R(x, y)$ be a binary relation on $\mathbb{N}$, such that $R(x, y)$ is true exactly when $2x \equiv y \mod 10$.

   (a) (2 points) Is the relation $R(x, y)$ *reflexive*?

   ◯ Yes √ **No**

   (b) (2 points) Is the relation $R(x, y)$ *symmetric*?

   ◯ Yes √ **No**

   (c) (2 points) Is the relation $R(x, y)$ *transitive*?

   ◯ Yes √ **No**

   (d) (6 points) Justify your answer to part (c) in a few short sentences.

   Solution:
   $R(1, 2)$ holds, and $R(2, 4)$ holds, but $R(1, 4)$ does not hold. So transitivity fails.

7. Together we are going to prove the following:

   Theorem. For any sets A, B (considered as subsets of universe U), $A \cap B = (A \cup B) \setminus (\overline{A} \cup \overline{B})$.

   Proof. We proceed by the set-element method. Fix $x : U$. We will show $x \in A \cap B \iff x \in (A \cup B) \setminus (\overline{A} \cup \overline{B})$.

   To show the first implication:

   1. Suppose $x \in A \cap B$.
   2. This means that $x \in A \wedge x \in B$.
   3. From that we see that $x \in A \vee x \in B$, and so $x \in A \cup B$.
   4. Since $x \in A$ we know $x \notin \overline{A}$, and similarly $x \notin \overline{B}$.
   5. So $x \notin \overline{A} \cup \overline{B}$.
   6. Since $x \in A \cup B$ but $x \notin \overline{A} \cup \overline{B}$, we conclude that $x \in (A \cup B) \setminus (\overline{A} \cup \overline{B})$.

   To show the second implication . . . : this is your job in part (c)!

   (a) (3 points) Consider the proof state after step 1. What hypotheses are in context? What is the goal?

   Solution: Hypotheses: $x \in A \cap B$. Goal: $x \in (A \cup B) \setminus (\overline{A} \cup \overline{B})$

   (b) (3 points) I have skipped some small details between steps 2 and 3. What proof rules did I need to use to make this inference?

   Solution: And elimination, or introduction

(c) (6 points) Complete the proof by showing the inclusion in the opposite direction.

Solution: To show the second implication, suppose $x \in (A \cup B) \setminus (\overline{A} \cup \overline{B})$. We aim to show $x \in A \cap B$.

Our hypothesis means that $x \in A \cup B$ but $x \notin \overline{A} \cup \overline{B}$. From the latter, we see that $\neg(x \notin A \vee x \notin B)$; by de Morgan, $x \in A \wedge x \in B$, so it is in the intersection.