Final Exam Reference Sheet

You do not need to cite any rules found here throughout your work. This is not comprehensive.

Logic

- **Disjunctive Normal Form:** a disjunction of conjunctions. EX: $(P \wedge Q) \vee (S \wedge R)$
- **Conjunctive Normal Form:** a conjunction of disjunctions. EX: $(P \vee Q) \wedge (S \vee R)$

Set Identities

1. **Set Difference Law:** For all sets A and B, $A \setminus B = A \cap \overline{B}$
2. **Distributive Law:** For all sets A, B, and C
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
3. **Identity Law:** For all sets A,
$$A \cup \emptyset = A, A \cap U = A$$
4. **De Morgan's Law:** For all sets A and B,
$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Relations and Functions

1. A **bijective** function is one that is **injective** and **surjective**.
   (a) A function $f : A \to B$ is **injective** if, for all $a, b \in A$, $f(a) = f(b) \to a = b$.
   (b) A function $f : A \to B$ is **surjective** if, for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Number Theory

Definition 1: We say that a *divides* b, denoted $a \mid b$, when $b = ka$ for some $k \in \mathbb{Z}$.

Definition 2: We say that a is *congruent* to b mod m, denoted $a \equiv b \pmod{m}$, if $m \mid (b - a)$.

Properties of Congruence Relations

For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{m}$, $a + c \equiv b + c \pmod{m}$ for any $c \in \mathbb{Z}$.

For $a, b \in \mathbb{Z}$, if $a \equiv b \pmod{m}$, $a \times c \equiv b \times c \pmod{m}$ for any $c \in \mathbb{Z}$.

Counting

The **binomial coefficient**, also called n *choose* k, is defined to be, for $n \geq k$ and $n, k \in \mathbb{N}$,

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Stars and Bars: The number of ways to distribute m identical objects among n distinct groups is

$$\binom{m+n-1}{n-1}.$$

Probability

Inclusion-Exclusion: For events A and B, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

The **conditional probability** $\Pr(A|B)$ is the probability that A happened given that we know B did. It is defined as

$$\Pr(A|B) \stackrel{\text{def}}{=} \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B|A) \cdot \Pr(A)}{\Pr(B)} \quad \text{(Bayes' Rule)}$$

A is **independent** of B if $\Pr(A|B) = \Pr(A)$, or if $\Pr(B) = 0$.

The **expected value** (or just expectation) of a random variable is a probability-weighted average of its values. The expected value of a random variable $R : \Omega \to S$ is:

$$\mathbb{E}[R] = \sum_{\omega \in S} R(\omega) \Pr(\omega)$$

Variance

Definition. The *variance* $\mathrm{Var}[R]$ of a random variable R is defined to be

$$\mathbb{E}[(R - \mathbb{E}[R])^2] \quad \textit{or equivalently} \quad \mathbb{E}[R^2] - (\mathbb{E}[R])^2.$$

Markov's Inequality: If R is a nonnegative random variable, then for all $x > 0$,

$$\Pr(R \geq x) \leq \frac{\mathbb{E}[R]}{x}.$$

Chebyshev's Inequality: Let R be a random variable and $x \in \mathbb{R}^+$. Then

$$\Pr\left(|R - \mathbb{E}\left[R\right]| \geq x\right) \leq \frac{\mathrm{Var}[R]}{x^2}.$$